ECE 172A: Introduction to Image Processing
Analog Images: Part I

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Outline

- Images as Functions
  - Vector-space formulation
  - Two-Dimensional **Systems**
- 2D Fourier Transform
  - Properties
  - Dirac Impulse, etc.
- Characterization of **LSI** Systems
  - Multidimensional **Convolution**
  - Modeling of Optical Systems
  - Examples of **Impulse Responses**

Images as Functions

- Analog = Continuously-Defined Image Representation
  - Images are functions of **two** real variables
- Vector-Space Formulation
  - All images are "points" in a vector space
- Vector Space of Finite-Energy Images
  - Mathematical framework for image representations
- Two-Dimensional Systems
- Linear, Shift-Invariant (LSI) Systems
  - Fundamental tool to "process" images

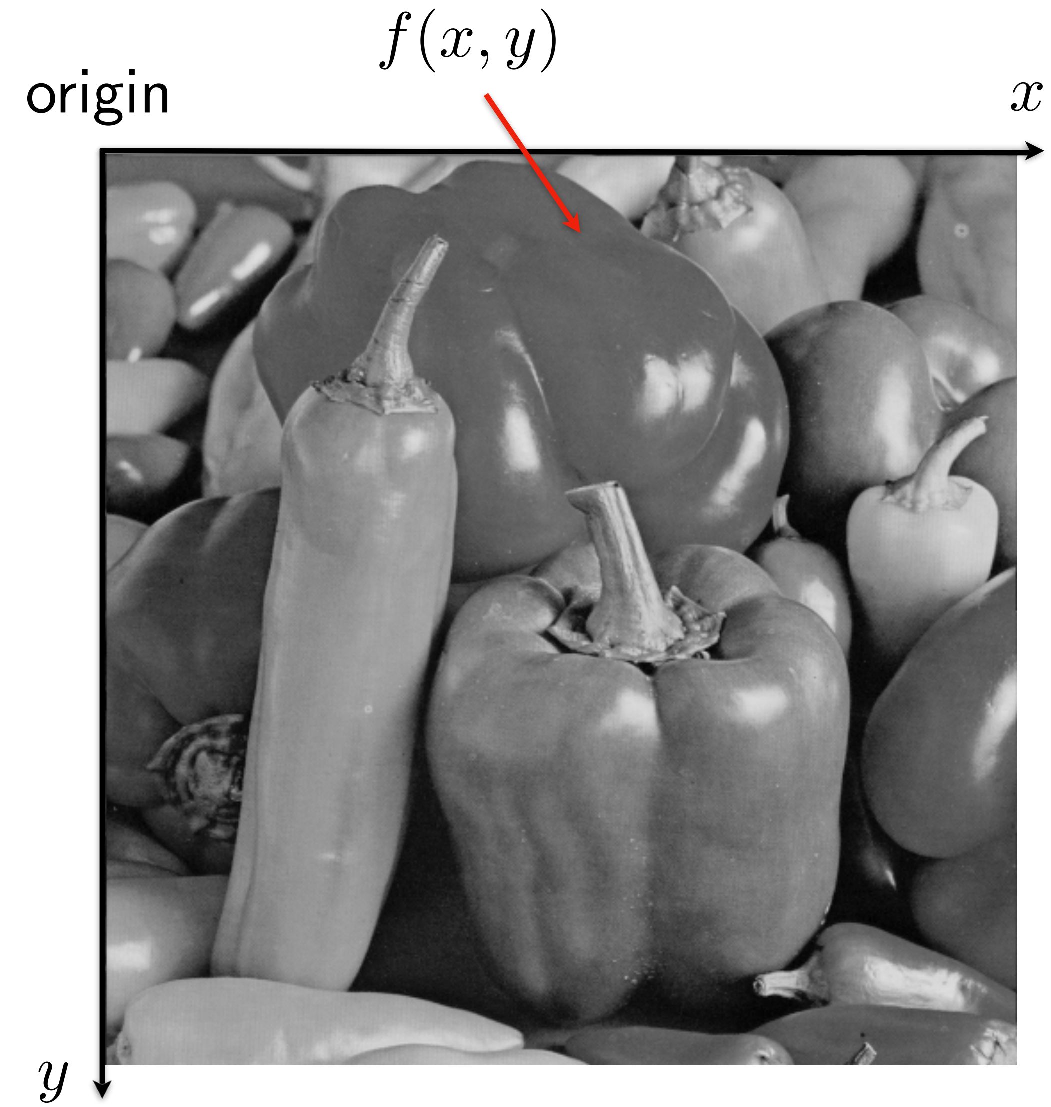
Analog Image Representation

Analog image

2D light intensity function: $f(x, y)$

- (x, y) are the spatial coordinates
- The output $f(x, y)$ is the **brightness** (or grayscale level) at (x, y)

$$f : \mathbb{R}^2 \to \mathbb{R}$$



Vector-Space Formulation

What is a vector space?

Definition: A vector space is a set $\mathcal{H}$ where, for every $f, g, h \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{R}$, we have that

- **Associativity:** $f + (g + h) = (f + g) + h$
- **Commutativity:** $f + g = g + f$
- **Identity:** There exists $0 \in \mathcal{H}$ such that $f + 0 = f$
- **Inverse:** There exists $-f \in \mathcal{H}$ such that $f + (-f) = 0$
- **Compatibility With Scalar Multiplication:** $\alpha(\beta f) = (\alpha\beta) f$
- **Multiplication With Scalar Identity** $1 f = f$ for $1 \in \mathbb{R}$
- **Distributivity I:** $\alpha(f + g) = \alpha f + \alpha g$
- **Distributivity II:** $(\alpha + \beta) f = \alpha f + \beta f$

Vector Space of Images

Do images (functions that map $\mathbb{R}^2 \rightarrow \mathbb{R}$) form a vector space?

- **Associativity:** $f + (g + h) = (f + g) + h$ ✓
- **Commutativity:** $f + g = g + f$ ✓
- **Identity:** There exists $0 \in \mathcal{H}$ such that $f + 0 = f$ ✓
- **Inverse:** There exists $-f \in \mathcal{H}$ such that $f + (-f) = 0$ ✓
- **Compatibility With Scalar Multiplication:** $\alpha(\beta f) = (\alpha\beta) f$ ✓
- **Multiplication With Scalar Identity** $1f = f$ for $1 \in \mathbb{R}$ ✓
- **Distributivity I:** $\alpha(f + g) = \alpha f + \alpha g$ ✓
- **Distributivity II:** $(\alpha + \beta) f = \alpha f + \beta f$ ✓

$\mathrm{zero}(x, y) = 0$ for all $(x, y) \in \mathbb{R}^2$

Yes, images form a vector space

Vector Space of Finite-Energy Images

Definition: The **energy** of an image $f : \mathbb{R}^2 \to \mathbb{R}$ is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2 \, \mathrm{d}x\mathrm{d}y$$

Definition: The **vector space** of finite-energy images is denoted $L^2(\mathbb{R}^2)$

$f \in L^2(\mathbb{R}^2)$ **if and only if** its energy is $< \infty$

$$\|f\|_{L^2}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2 \, \mathrm{d}x\mathrm{d}y$$

"squared L^2-norm of f or energy of f"

measures the "size" of f

Recall: Given a vector $\boldsymbol{x} \in \mathbb{R}^N$

$$\|\boldsymbol{x}\|_2^2 = \sum_{n=1}^{N} |x_n|^2$$

Inner Product of Finite-Energy Images

Recall: Given vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$, their inner product is

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\mathsf{T} \boldsymbol{y} = \sum_{n=1}^{N} x_n y_n$$

Definition: The **inner product** of $f, g \in L^2(\mathbb{R}^2)$ is

conjugate if complex valued

$$\langle f, g \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) g^*(x, y) \, \mathrm{d}x \mathrm{d}y$$

Observation: The norm is **induced by** the inner product

$$\|f\|_{L^2}^2 = \langle f, f \rangle$$

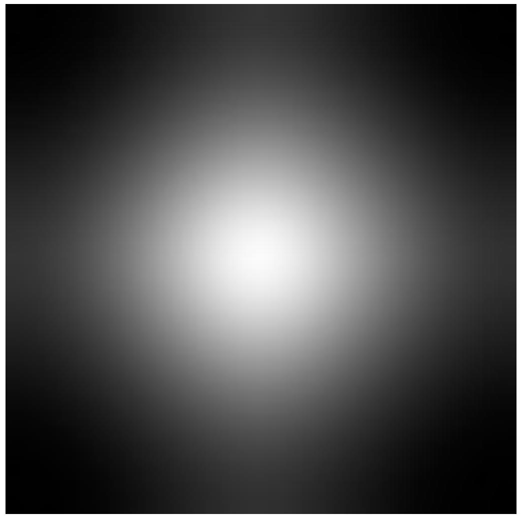
$$L^2(\mathbb{R}^2) = \{ f(x, y) \ : \ \|f\|_{L^2}^2 = \langle f, f \rangle < \infty \}$$

Examples of Finite-Energy Images

- 2D Gaussian

$$g(x,y) = \frac{1}{2\pi} \exp\left(-\frac{x^2+y^2}{2}\right)$$

$$g \in L^2(\mathbb{R}^2)$$



- Finite support $\Omega \subset \mathbb{R}^2$ and bounded images

$$\begin{cases} f(x,y) = 0, & \text{for all } (x,y) \notin \Omega \\ |f(x,y)| < C, & \text{for all } (x,y) \in \mathbb{R}^2 \end{cases}$$

$$\Omega = [0,1] \times [0,1] = [0,1]^2$$

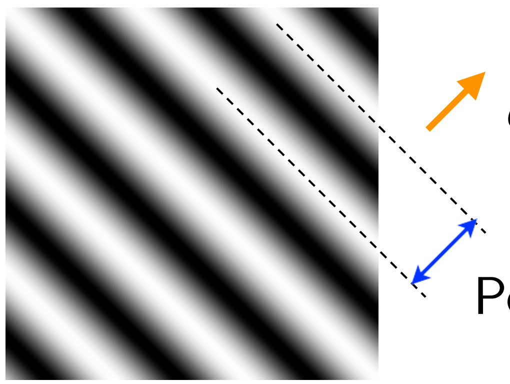
Exercise: Show that $f \in L^2(\mathbb{R}^2)$

$$\begin{aligned} \|f\|_{L^2}^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2 \, \mathrm{d}x\mathrm{d}y \\ &= \iint_{\Omega} |f(x,y)|^2 \, \mathrm{d}x\mathrm{d}y \\ &< \iint_{\Omega} C^2 \, \mathrm{d}x\mathrm{d}y \\ &= C^2 \operatorname{vol}(\Omega) < \infty \end{aligned}$$

Plane Waves

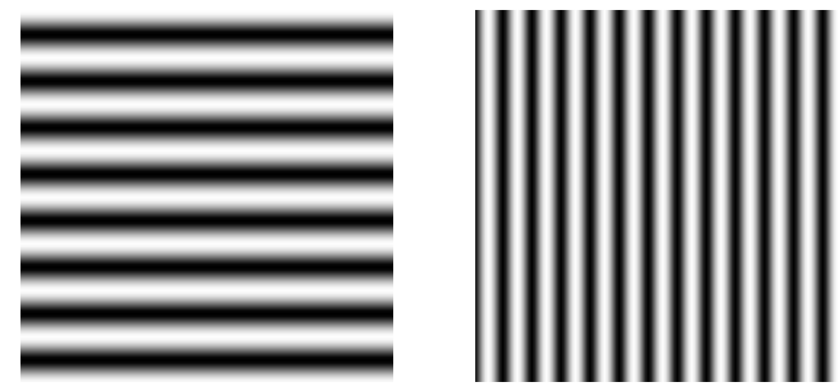
- Sinusoidal gratings

$$s(x,y) = A\ \cos(\omega_1 x + \omega_2 y + \phi)$$



Wave vector $\boldsymbol{\omega} = (\omega_1, \omega_2)$

Period: $T = \dfrac{2\pi}{\sqrt{\omega_1^2 + \omega_2^2}}$



Does s have finite energy?

No, $s \notin L^2(\mathbb{R}^2)$

However, $s(x,y) \cdot w(x,y) \in L^2(\mathbb{R}^2)$

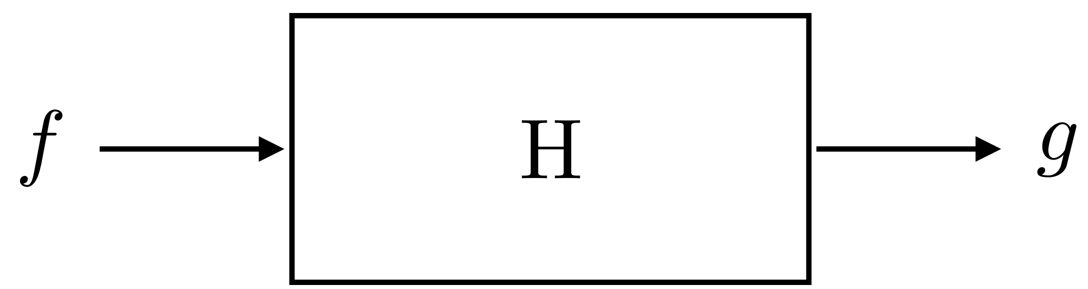
$w(x,y)$ is a finite-support and bounded **window function**

Example:

$$\begin{cases} w(x,y) = 1, & (x,y) \in [0,1]^2 \\ w(x,y) = 0, & \text{else} \end{cases}$$

Two-Dimensional Systems

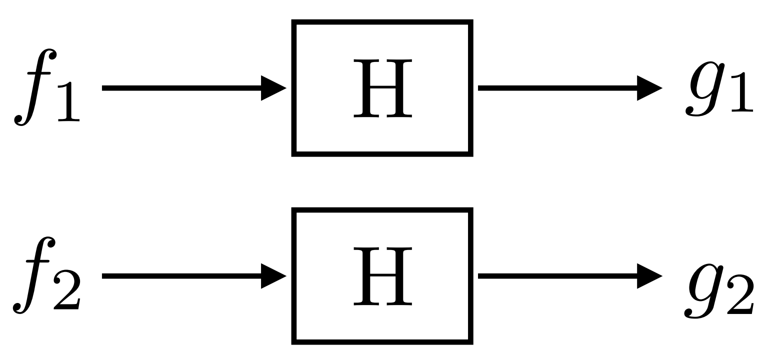
- Mapping from one image to another

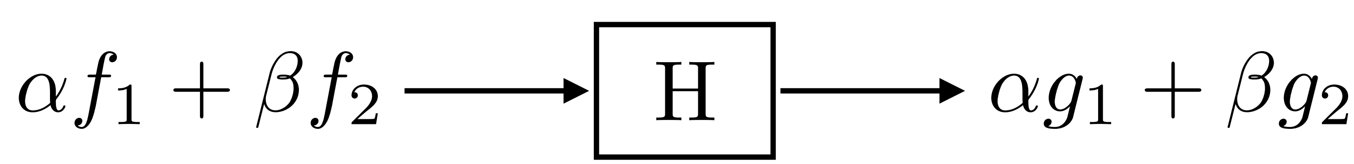


$$H : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$
$$g = H\{f\}$$

- The most important systems are **linear systems**

$\forall \alpha, \beta \in \mathbb{R}$





$$H\{\alpha f_1 + \beta f_2\} = \alpha H\{f_1\} + \beta H\{f_2\}$$

Linearity Practice

- (Partial) derivative operators are linear or nonlinear? **Linear**

$$\mathrm{H}_1\{f\} = \frac{\partial f(x,y)}{\partial x} \quad \text{and} \quad \mathrm{H}_2\{f\} = \frac{\partial f(x,y)}{\partial y}$$

- The following operator is linear or nonlinear? **Linear**

$$\mathrm{H}_3\{f\}(x,y) = f(x^2 + x + 1, y - \sqrt{y})$$

- Geometric operators are linear or nonlinear? **Linear**

$$\mathrm{H}_4\{f\}(x,y) = f(G_1(x,y), G_2(x,y))$$

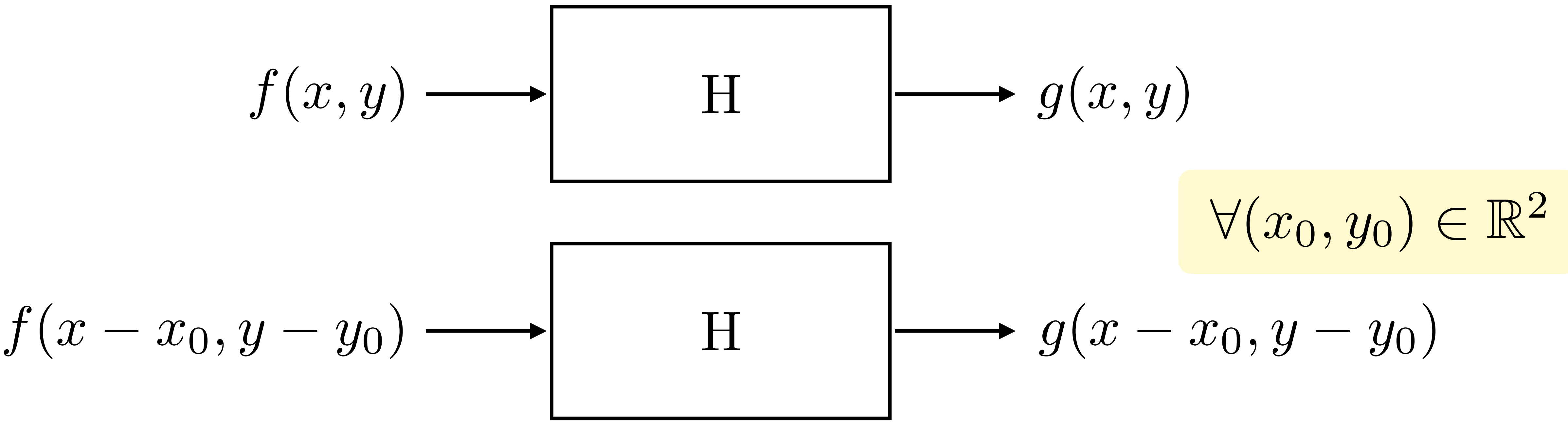
  where $G_1(x,y)$ and $G_2(x,y)$ are arbitrary (nonlinear) transformations.

- The thresholding operator is linear or nonlinear? **Nonlinear**

$$\mathrm{H}_5\{f\}(x,y) = \begin{cases} 1, & |f(x,y)| \geq T_0 \\ 0, & \text{else} \end{cases}$$

Linear, Shift-Invariant Systems (LSI)

Definition: A linear system H is **shift-invariant** if and only if shifted inputs correspond to shifted outputs.



$$\mathrm{H}\{f(x-x_0, y-y_0)\} = \mathrm{H}\{f\}(x-x_0, y-y_0)$$

- LSI systems model most physical imaging devices

"impulse response"

LSI $=$ realized by **convolution**: $\mathrm{H}\{f\}(x,y) = (h * f)(x,y)$

2D Fourier Transform

- Definition
- Separability
- Properties
- Dirac impulse
- Dirac related Fourier transforms
- Application: finding the orientation
- Importance of the phase

2D Fourier Transform: Definition

- 2D Fourier transform: $\hat{f}(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{-\mathrm{j}(\omega_1 x + \omega_2 y)}\,\mathrm{d}x\,\mathrm{d}y$
- Inverse Fourier transform: $f(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\omega_1, \omega_2)\mathrm{e}^{\mathrm{j}(\omega_1 x + \omega_2 y)}\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2$

Vector notation:

Spatial variables: $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$

Frequency variables: $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2$

$\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x} = \omega_1 x + \omega_2 y$

$$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}$$

$$\Big\updownarrow \mathcal{F}$$

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega})\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{\omega}$$

Plancherel, Parseval, and Finite-Energy

- Fourier analysis on $L^2(\mathbb{R}^2)$ (Plancherel)

$$f \in L^2(\mathbb{R}^2) \quad \text{if and only if} \quad \hat{f} \in L^2(\mathbb{R}^2)$$

- Parseval's formula for $f, g \in L^2(\mathbb{R}^2)$

$$\langle f, g \rangle = \frac{1}{(2\pi)^2} \langle \hat{f}, \hat{g} \rangle$$

- Plancherel's theorem for $f \in L^2(\mathbb{R}^2)$

$$\|f\|_{L^2}^2 = \frac{1}{(2\pi)^2} \|\hat{f}\|_{L^2}^2$$

What does this mean?

Fourier analysis is well-matched to finite-energy functions

Separability

- Separability of complex exponential: $\mathrm{e}^{-\mathrm{j}(\omega_1 x+\omega_2 y)} = \mathrm{e}^{-\mathrm{j}\omega_1 x}\mathrm{e}^{-\mathrm{j}\omega_2 y}$

2D Fourier transform = sequence of two 1D Fourier transforms

Fourier in x then y or Fourier in y then x

Exercise: Show that this is true.

1D Fourier transform in x: $\displaystyle\int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{-\mathrm{j}\omega_1 x}\,\mathrm{d}x = \hat{f}_y(\omega_1)$

1D Fourier transform in y: $\displaystyle\int_{-\infty}^{\infty} \hat{f}_y(\omega_1)\mathrm{e}^{-\mathrm{j}\omega_2 y}\,\mathrm{d}y$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{-\mathrm{j}\omega_1 x}\,\mathrm{d}x\,\mathrm{e}^{-\mathrm{j}\omega_2 y}\,\mathrm{d}y$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{-\mathrm{j}\omega_1 x}\mathrm{e}^{-\mathrm{j}\omega_2 y}\,\mathrm{d}x\,\mathrm{d}y = \hat{f}(\omega_1,\omega_2)$$

2D Fourier transform inherits most properties from 1D Fourier transform!

Separability (cont'd)

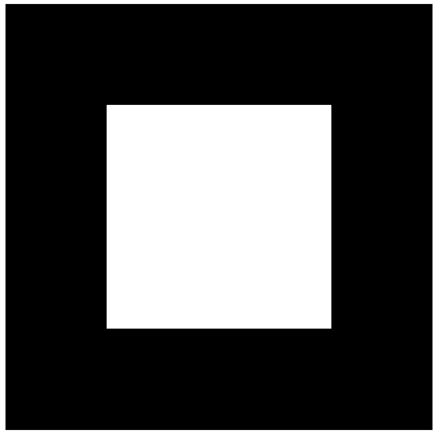
Definition: $f(x, y)$ is called separable if $f(x, y) = f_1(x) f_2(y)$ for some $f_1(x)$ and $f_2(y)$.

Exercise: For separable functions, show that $\hat{f}(\omega_1, \omega_2) = \hat{f}_1(\omega_1)\hat{f}_2(\omega_2)$.

What is an example of a separable function?

$$f(x, y) = \begin{cases} 1, & \text{if } (x, y) \in [0, 1]^2 \\ 0, & \text{else} \end{cases}$$

"box" or "rect" function

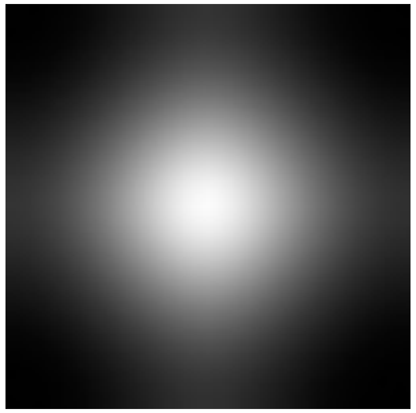


$$f_1(x) = \begin{cases} 1, & \text{if } x \in [0, 1] \\ 0, & \text{else} \end{cases}$$

$$f(x, y) = f_1(x) f_1(y)$$

Separability (cont'd)

- 2D Gaussian



$$g(x,y) = \exp\left(-\frac{x^2+y^2}{2}\right)$$
$$= \mathrm{e}^{-x^2/2}\mathrm{e}^{-y^2/2}$$

$$\Longrightarrow \quad \hat{g}(\omega_1,\omega_2) = \hat{f}(\omega_1)\hat{f}(\omega_2)$$ where $f(x) = \mathrm{e}^{-x^2/2}$

$\updownarrow \mathcal{F}$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{j}\omega x}\,\mathrm{d}x = \sqrt{2\pi}\mathrm{e}^{-\omega^2/2}$$

$$= \sqrt{2\pi}\mathrm{e}^{-\omega_1^2/2}\cdot\sqrt{2\pi}\mathrm{e}^{-\omega_2^2/2} = 2\pi\exp\left(-\frac{\omega_1^2+\omega_2^2}{2}\right)$$

Fourier transform of a Gaussian is a Gaussian (just like 1D)

Fourier Properties

- Duality: $$\hat{f}(\boldsymbol{x}) \stackrel{\mathcal{F}}{\longleftrightarrow} (2\pi)^2 f(-\boldsymbol{\omega})$$
- Symmetry: $$f(\boldsymbol{x}) \text{ real} \Leftrightarrow \hat{f}^*(\boldsymbol{\omega}) = \hat{f}(-\boldsymbol{\omega})$$
- Energy-Preservation: $$\|f\|_{L_2}^2 = (2\pi)^{-2}\|\hat{f}\|_{L^2}^2$$
- Shift: $$f(\boldsymbol{x}-\boldsymbol{x}_0) \stackrel{\mathcal{F}}{\longleftrightarrow} \mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{x}_0}\hat{f}(\boldsymbol{\omega})$$
- Modulation: $$\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}_0^{\mathsf{T}}\boldsymbol{x}} f(\boldsymbol{x}) \stackrel{\mathcal{F}}{\longleftrightarrow} \hat{f}(\boldsymbol{\omega}-\boldsymbol{\omega}_0)$$
- Scaling: $$f(\boldsymbol{x}/\alpha) \stackrel{\mathcal{F}}{\longleftrightarrow} |\alpha|^2 \hat{f}(\alpha\boldsymbol{\omega})$$
- Affine Transformation: $$f(\mathbf{A}\boldsymbol{x}) \stackrel{\mathcal{F}}{\longleftrightarrow} |\det \mathbf{A}|^{-1}\hat{f}((\mathbf{A}^{-1})^{\mathsf{T}}\boldsymbol{\omega})$$
- Differentiation: $$\frac{\partial^n f(\boldsymbol{x})}{\partial x^n} \stackrel{\mathcal{F}}{\longleftrightarrow} (\mathrm{j}\omega_1)^n \hat{f}(\boldsymbol{\omega})$$ $$\frac{\partial^n f(\boldsymbol{x})}{\partial y^n} \stackrel{\mathcal{F}}{\longleftrightarrow} (\mathrm{j}\omega_2)^n \hat{f}(\boldsymbol{\omega})$$
- Moments: $$\mu_f^{m,n} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m y^n f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \mathrm{j}^{m+n} \left.\frac{\partial^{m+n}\hat{f}(\boldsymbol{\omega})}{\partial\omega_1^m \partial\omega_2^n}\right|_{\omega_1=0,\omega_2=0}$$

  In particular, $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \hat{f}(\mathbf{0}) = \hat{f}(0,0)$$

Dirac Impulse

- Recall the 1D Dirac impulse $\delta(x)$: $\langle f, \delta \rangle = \int_{-\infty}^{\infty} f(x)\delta(x)\,\mathrm{d}x = f(0)$
- Properties:

Normalized integral: $\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$

Fourier transform: $\delta(x) \stackrel{\mathcal{F}}{\longleftrightarrow} 1$

Convolution: $(g * \delta)(x) = \int_{-\infty}^{\infty} \delta(u)g(x-u)\,\mathrm{d}u = g(x)$

Exercise: Prove these three properties using the definition.

Normalized integral: $f(x) = 1$

Fourier transform: $f(x) = \mathrm{e}^{\mathrm{j}\omega x}$

Convolution: $f(u) = g(x-u)$

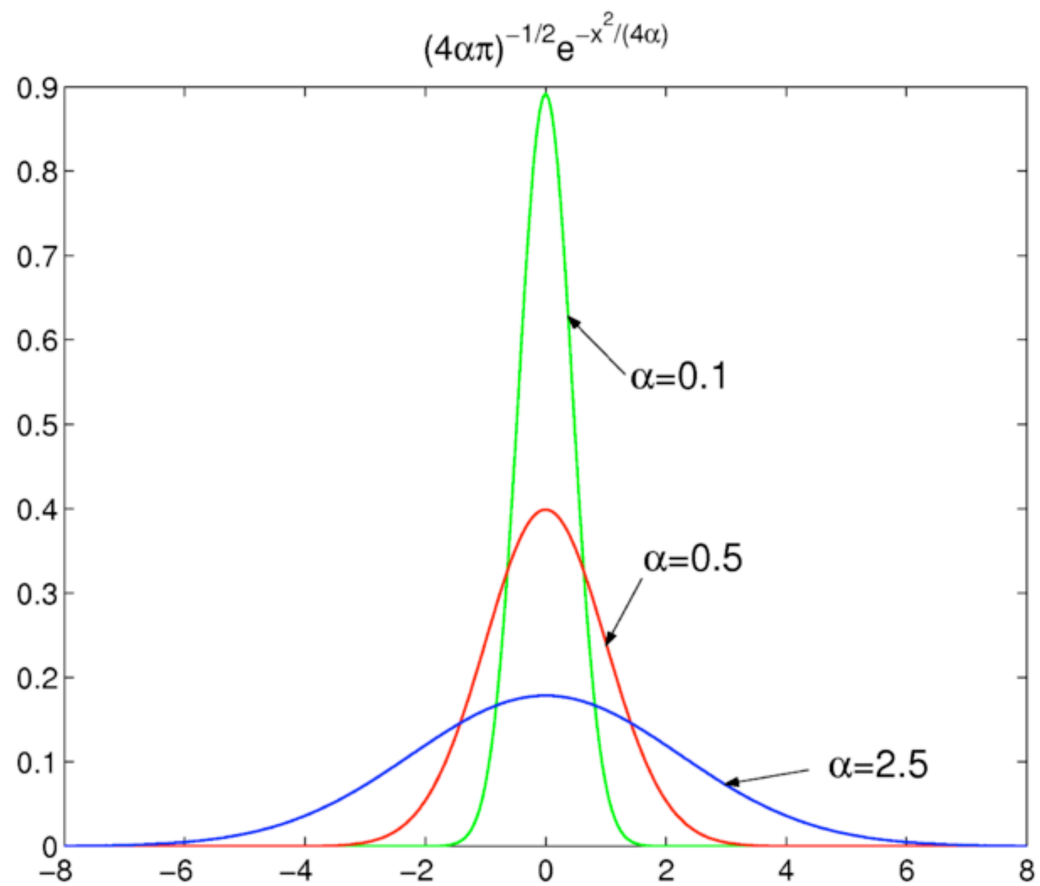
Explicit Construction of the Dirac Impulse

- Consider any window function $\varphi(x)$ such that $\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x = 1$
- Observe that $\int_{-\infty}^{\infty} \frac{1}{|\alpha|}\varphi\left(\frac{x}{\alpha}\right)\,\mathrm{d}x = 1$

"integral-preserving dilation/contraction"

- $\delta(x) = \lim_{\alpha \to 0} \left(\frac{1}{|\alpha|}\varphi\left(\frac{x}{\alpha}\right)\right)$

e.g., φ is a Gaussian, rectangle, triangle, etc.



2D Dirac Impulse

- A reasonable definition: $\langle f, \delta \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\delta(x,y)\,\mathrm{d}x = f(0,0)$

What could give us this?

$$\delta(x,y) = \delta(x)\delta(y) \overset{\mathcal{F}}{\longleftrightarrow} 1 \cdot 1 = 1$$

The Dirac impulse is **separable**!

Exercise: Prove that this is the 2D Dirac impulse.

- Properties:
  - Normalized integral: $\int_{\mathbb{R}^2} \delta(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$
  - Fourier transform: $\delta(\boldsymbol{x}) \overset{\mathcal{F}}{\longleftrightarrow} 1$
  - Multiplication: $f(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{x}_0) = f(\boldsymbol{x}_0)\delta(\boldsymbol{x} - \boldsymbol{x}_0)$
  - Sampling: $\langle f(\boldsymbol{x}), \delta(\boldsymbol{x} - \boldsymbol{x}_0) \rangle = \int_{\mathbb{R}^2} f(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{x}_0)\,\mathrm{d}\boldsymbol{x} = f(\boldsymbol{x}_0)$
  - Convolution: $(f * \delta)(\boldsymbol{x}) = f(\boldsymbol{x})$
  - Scaling: $\delta(\boldsymbol{x}/\alpha) = |\alpha|^2 \delta(\boldsymbol{x})$

These properties are deduced from the 1D Dirac properties.

Dirac-Related Fourier Transforms

- Constant

One-dimensional: $1 \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega x}\,\mathrm{d}x = ???$

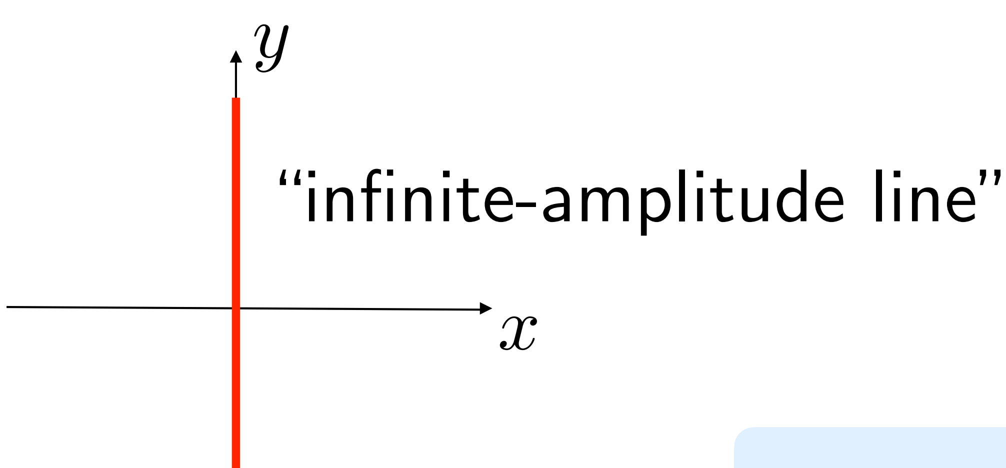
$$= \lim_{A\to\infty} \int_{-A}^{A} \mathrm{e}^{-\mathrm{j}\omega x}\,\mathrm{d}x = 2\pi\,\delta(\omega)$$

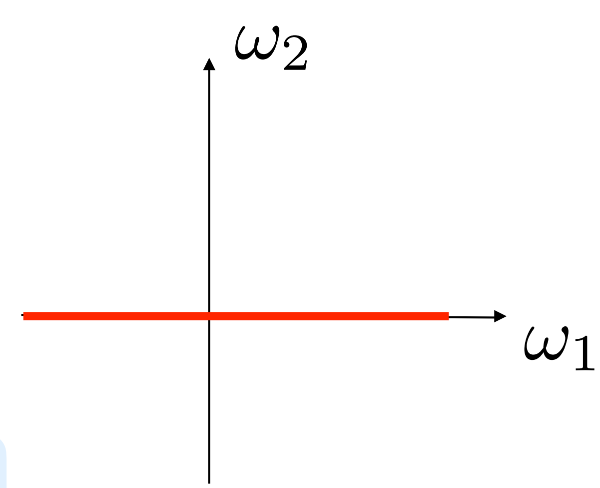
(or by duality)

Two-dimensional: $1 \overset{\mathcal{F}}{\longleftrightarrow} (2\pi)^2\delta(\boldsymbol{\omega}) = (2\pi)^2\delta(\omega_1,\omega_2)$

- Dirac line (or “ideal” line)

$$f(x,y) = \delta(x)\cdot 1 = f_1(x)f_2(y) \overset{\mathcal{F}}{\longleftrightarrow} \hat{f}(\omega_1,\omega_2) = \hat{f}_1(\omega_1)\hat{f}_2(\omega_2) = 1\cdot 2\pi\delta(\omega_2)$$





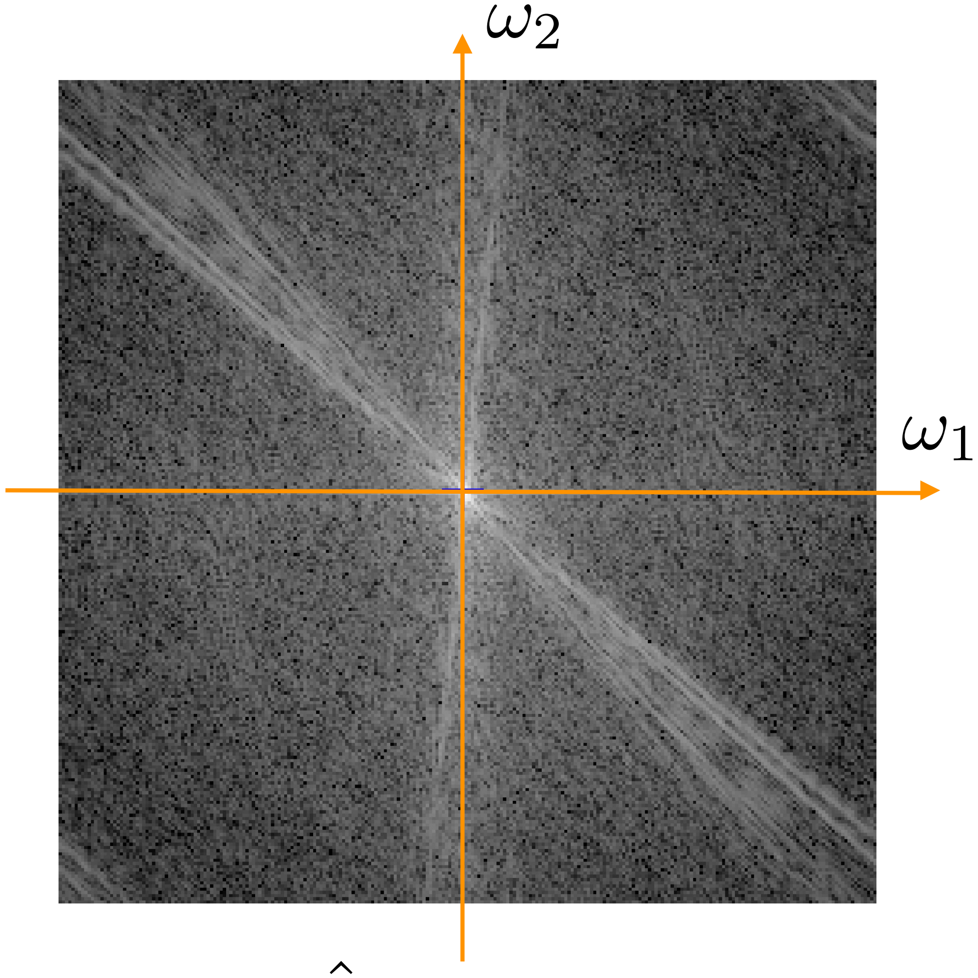
What does this mean?

Example

Spatial Domain

$f(x, y)$

Fourier Domain



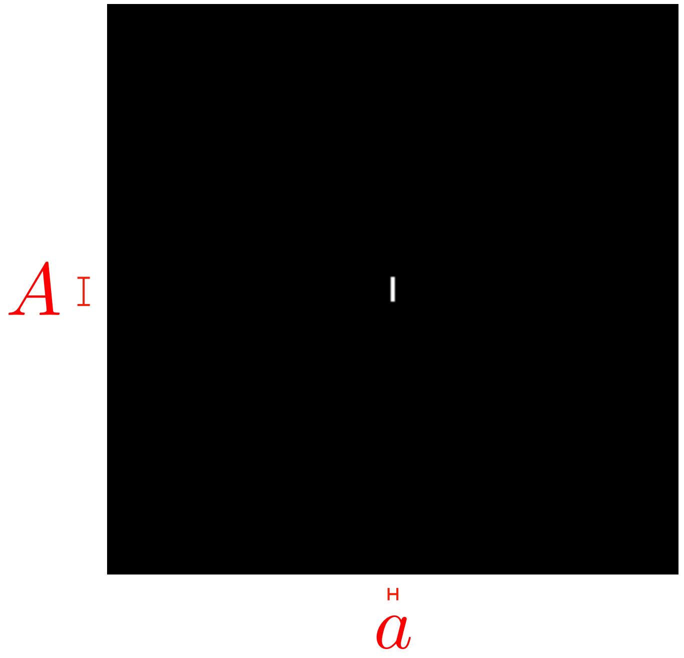
$\hat{f}(\omega_1, \omega_2)$

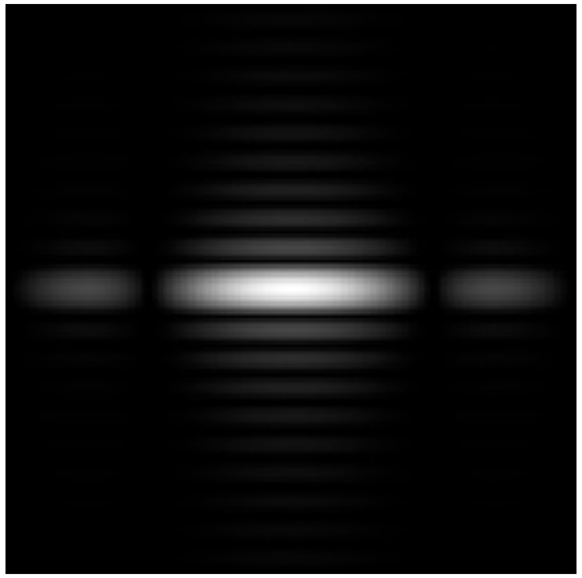
What are these two sets of lines?

More-Realistic Line Model

- Rectangular shape

$$f(x,y) = \mathrm{rect}(x/a)\,\mathrm{rect}(y/A) \overset{\mathcal{F}}{\longleftrightarrow} |a|\mathrm{sinc}\left(\frac{a\omega_1}{2\pi}\right)\,|A|\mathrm{sinc}\left(\frac{A\omega_2}{2\pi}\right)$$





Reminder:

$$\mathrm{rect}(x) = \begin{cases} 1, & \text{if } x \in [-1/2, 1/2] \\ 0, & \text{else} \end{cases} \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{\sin(\omega/2)}{\omega/2}$$