ECE 172A: Introduction to Image Processing
Analog Images: Part II

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Outline

- Images as Functions ✓
  - Vector-space formulation
  - Two-Dimensional **Systems**
- 2D Fourier Transform
  - Properties
  - Dirac Impulse, etc.
- Characterization of **LSI** Systems
  - Multidimensional **Convolution**
  - Modeling of Optical Systems
  - Examples of **Impulse Responses**

2D Fourier Transform

- Definition ✓
- Separability ✓
- Properties ✓
- Dirac impulse ✓
- Dirac related Fourier transforms ✓
- Application: finding the orientation
- Importance of the phase

Dirac-Related Fourier Transforms

- Constant

One-dimensional: $1 \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}\omega x} \, \mathrm{d}x = ???$

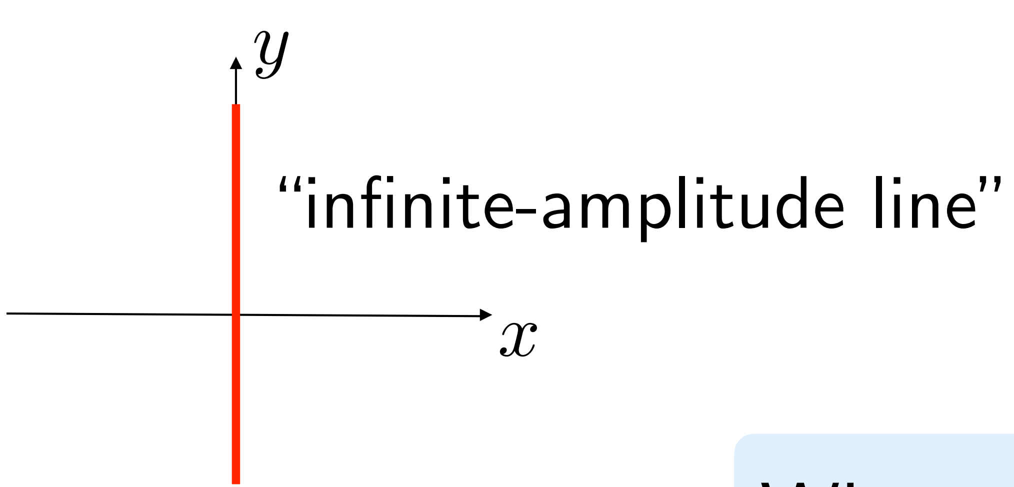
$$= \lim_{A \to \infty} \int_{-A}^{A} \mathrm{e}^{-\mathrm{j}\omega x} \, \mathrm{d}x = 2\pi \, \delta(\omega)$$

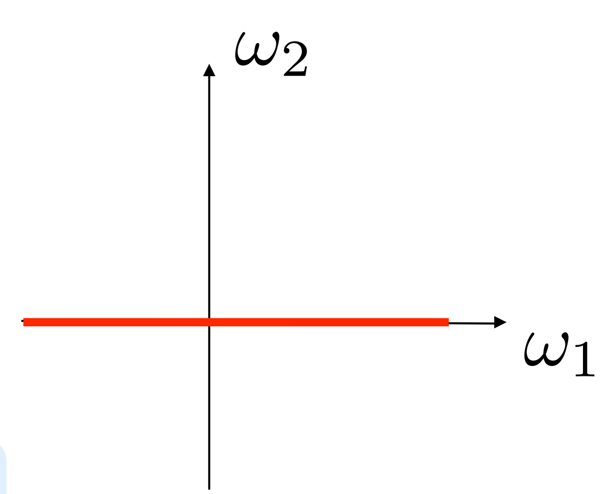
(or by duality)

Two-dimensional: $1 \overset{\mathcal{F}}{\longleftrightarrow} (2\pi)^2 \delta(\boldsymbol{\omega}) = (2\pi)^2 \delta(\omega_1, \omega_2)$

- Dirac line (or “ideal” line)

$$f(x, y) = \delta(x) \cdot 1 = f_1(x) f_2(y) \xleftrightarrow{\mathcal{F}} \hat{f}(\omega_1, \omega_2) = \hat{f}_1(\omega_1) \hat{f}_2(\omega_2) = 1 \cdot 2\pi\delta(\omega_2)$$





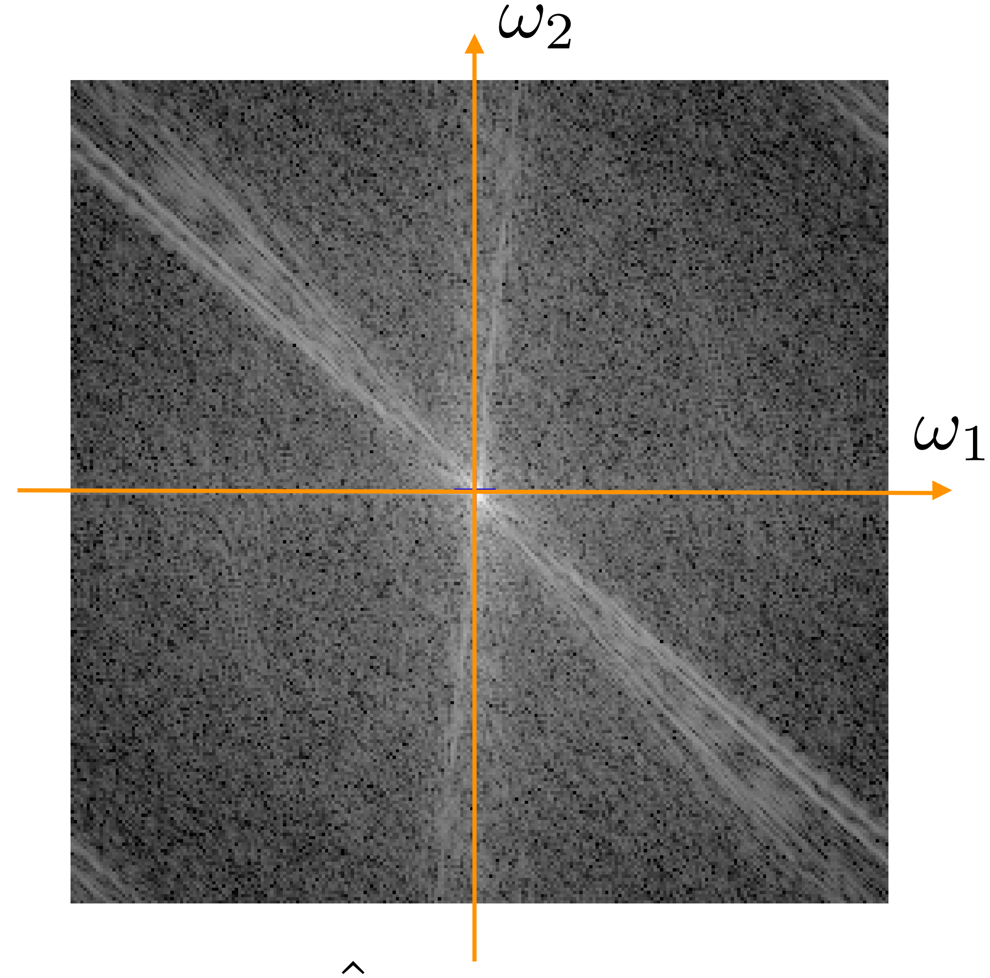
What does this mean?

Example

Spatial Domain

$f(x, y)$

Fourier Domain



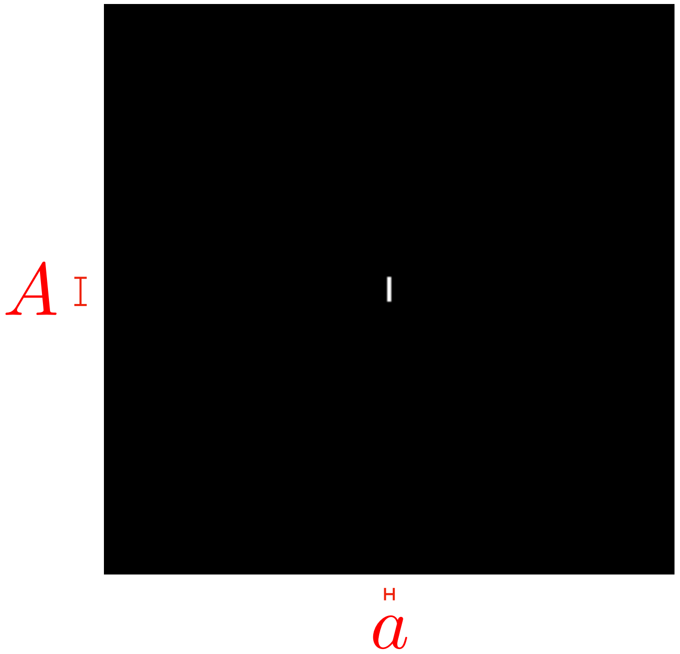
$\hat{f}(\omega_1, \omega_2)$

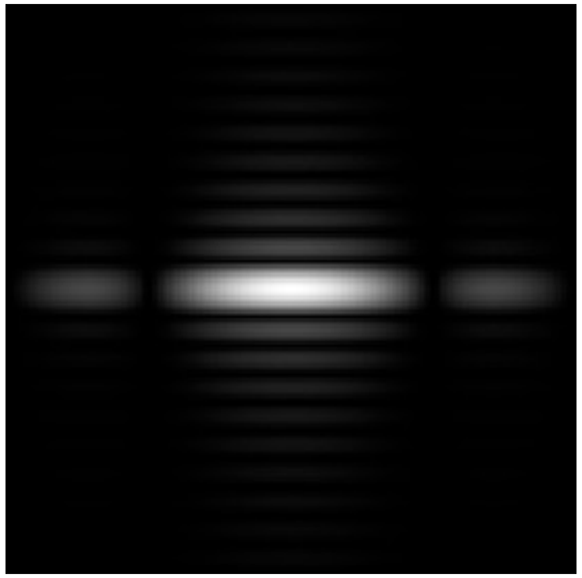
What are these two sets of lines?

More-Realistic Line Model

- Rectangular shape

$$f(x,y) = \mathrm{rect}(x/a)\,\mathrm{rect}(y/A) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad |a|\mathrm{sinc}\left(\frac{a\omega_1}{2\pi}\right)\,|A|\mathrm{sinc}\left(\frac{A\omega_2}{2\pi}\right)$$





Reminder:

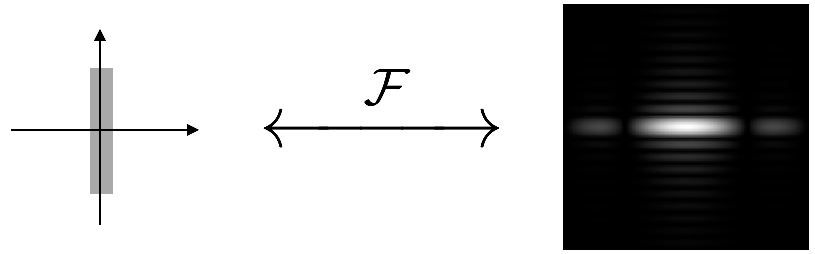
$$\mathrm{rect}(x) = \begin{cases} 1, & \text{if } x \in [-1/2, 1/2] \\ 0, & \text{else} \end{cases} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) = \frac{\sin(\omega/2)}{\omega/2}$$

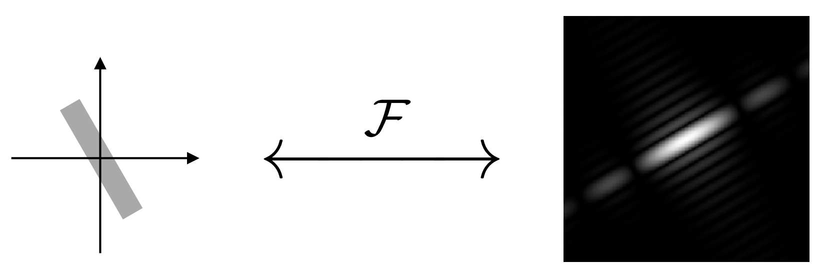
Application: Orientation Estimation

- **Problem:** Design a (real time?) system that can determine the orientation of a (linear) pattern placed at an arbitrary location in an image.

Any ideas?

- What do we know?





$$g_\theta(\boldsymbol{x}) = f(\mathbf{R}_\theta \boldsymbol{x})$$

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$g_\theta(\boldsymbol{x}) \quad \xleftrightarrow{\mathcal{F}} \quad \hat{f}(\mathbf{R}_\theta \boldsymbol{\omega})$$

We want to find the orientation in the Fourier domain with the **least spread**.

Problem Solution

- Compute the "Fourier inertia" matrix (second-moment matrix)

$$\mathbf{M} = \begin{bmatrix} \iint \omega_1^2 |\hat{f}(\boldsymbol{\omega})|^2 \, \mathrm{d}\omega_1 \, \mathrm{d}\omega_2 & \iint \omega_1 \omega_2 |\hat{f}(\boldsymbol{\omega})|^2 \, \mathrm{d}\omega_1 \, \mathrm{d}\omega_2 \\ \iint \omega_2 \omega_1 |\hat{f}(\boldsymbol{\omega})|^2 \, \mathrm{d}\omega_1 \, \mathrm{d}\omega_2 & \iint \omega_2^2 |\hat{f}(\boldsymbol{\omega})|^2 \, \mathrm{d}\omega_1 \, \mathrm{d}\omega_2 \end{bmatrix}$$

$$= \begin{bmatrix} \langle \mathrm{j}\omega_1 \hat{f}(\boldsymbol{\omega}), \mathrm{j}\omega_1 \hat{f}(\boldsymbol{\omega}) \rangle & \langle \mathrm{j}\omega_1 \hat{f}(\boldsymbol{\omega}), \mathrm{j}\omega_2 \hat{f}(\boldsymbol{\omega}) \rangle \\ \langle \mathrm{j}\omega_2 \hat{f}(\boldsymbol{\omega}), \mathrm{j}\omega_1 \hat{f}(\boldsymbol{\omega}) \rangle & \langle \mathrm{j}\omega_2 \hat{f}(\boldsymbol{\omega}), \mathrm{j}\omega_2 \hat{f}(\boldsymbol{\omega}) \rangle \end{bmatrix}$$

Second-order moments measure spread

$$= (2\pi)^2 \begin{bmatrix} \langle \partial_x f, \partial_x f \rangle & \langle \partial_x f, \partial_y f \rangle \\ \langle \partial_y f, \partial_x f \rangle & \langle \partial_y f, \partial_y f \rangle \end{bmatrix}$$ (fast algorithm via Parseval-Plancherel)

Which direction will have the least spread?

The direction of the **smallest eigenvalue**

Problem Solution (cont'd)

- Eigendecomposition of $\mathbf{M}$ gives us the **axes of inertia**

$$\lambda_1 \geq \lambda_2$$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1^\top \\ \boldsymbol{u}_2^\top \end{bmatrix} \mathbf{M} \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}$$

$\boldsymbol{u}_1$: eigenvector in the direction of the **long** axis
$\boldsymbol{u}_2$: eigenvector in the direction of the **short** axis

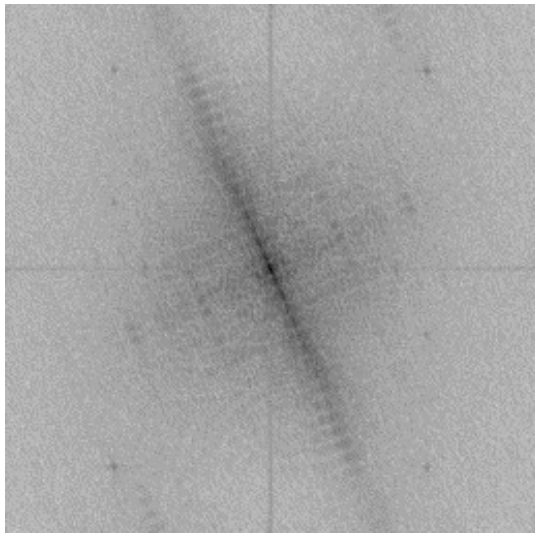
- Pipeline:
  1. Compute the Fourier inertia matrix $\mathbf{M}$ via the fast algorithm
  2. Compute the eigendecomposition of $\mathbf{M}$ and store $\boldsymbol{u}_2$
  3. Return the angle of $\boldsymbol{u}_2$
     * $\theta = \arctan \dfrac{u_{22}}{u_{21}}$

Orientation Estimation in Action

- Image 1:

  Measured angle: $25° \pm 2°$

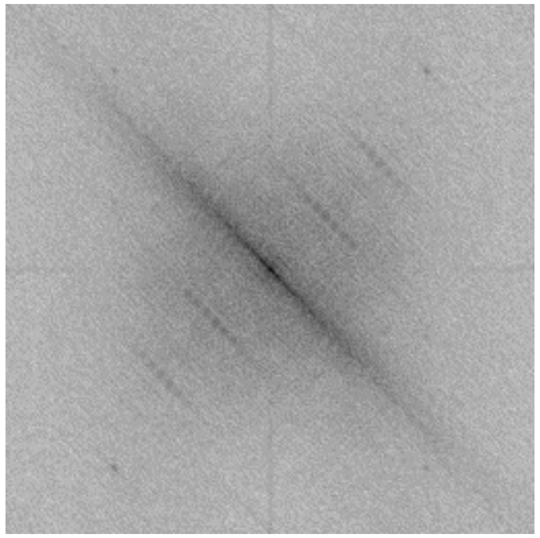
  Computed angle: $27°$



- Image 2:

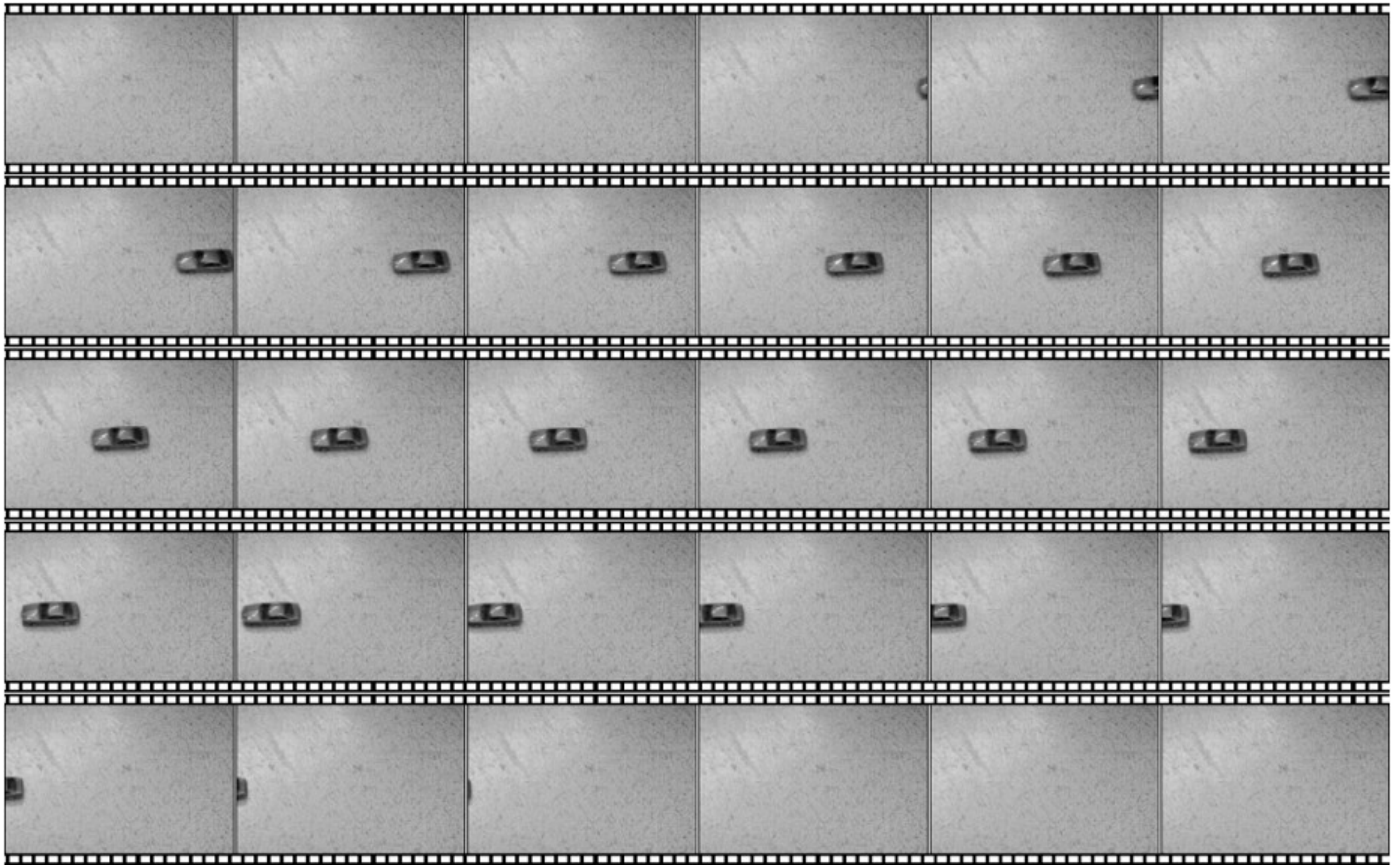
  Measured angle: $44° \pm 2°$

  Computed angle: $45.6°$



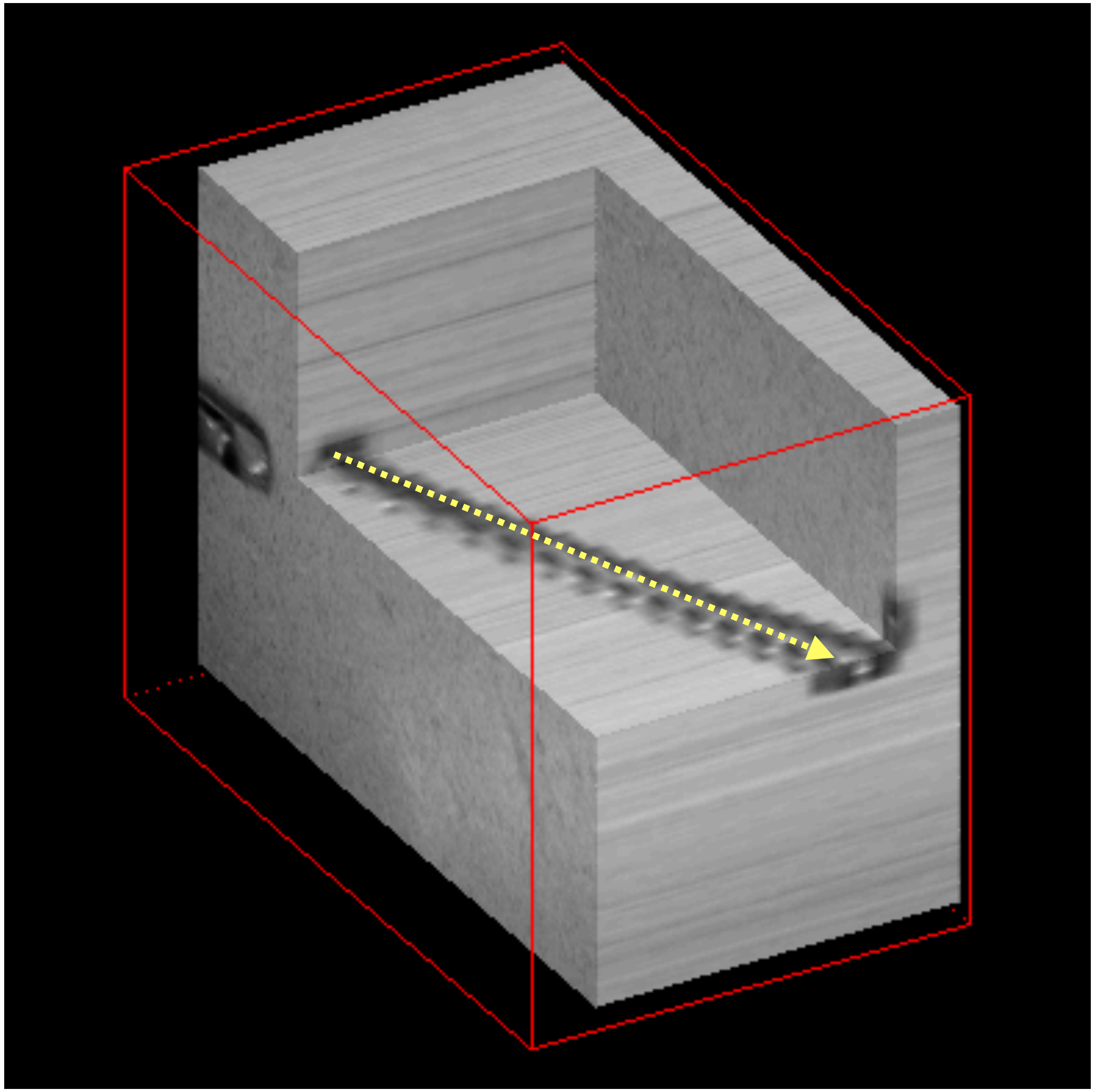
This is how (pre-deep-learning-based) machine vision worked

Video Example



How do we determine the direction of the car?

Video Example (cont'd)



What are the dimensions of the Fourier inertia matrix?

Magnitude and Phase Information

- The Fourier transform is **complex-valued**

$$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) \mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}} \,\mathrm{d}\boldsymbol{x} = \underbrace{|\hat{f}(\boldsymbol{\omega})|}_{\text{magnitude}} \exp\big(\mathrm{j}\underbrace{\phi(\boldsymbol{\omega})}_{\text{phase}}\big)$$

- Fourier magnitude

$$|\hat{f}(\boldsymbol{\omega})| = \left(\hat{f}(\boldsymbol{\omega})\, \hat{f}^*(\boldsymbol{\omega})\right) = \sqrt{(\Re \hat{f}(\boldsymbol{\omega}))^2 + (\Im \hat{f}(\boldsymbol{\omega}))^2}$$

- Fourier phase

$$\phi(\boldsymbol{\omega}) = \arg\left(\hat{f}(\boldsymbol{\omega})\right) = \arctan\left(\frac{\Im \hat{f}(\boldsymbol{\omega})}{\Re \hat{f}(\boldsymbol{\omega})}\right)$$

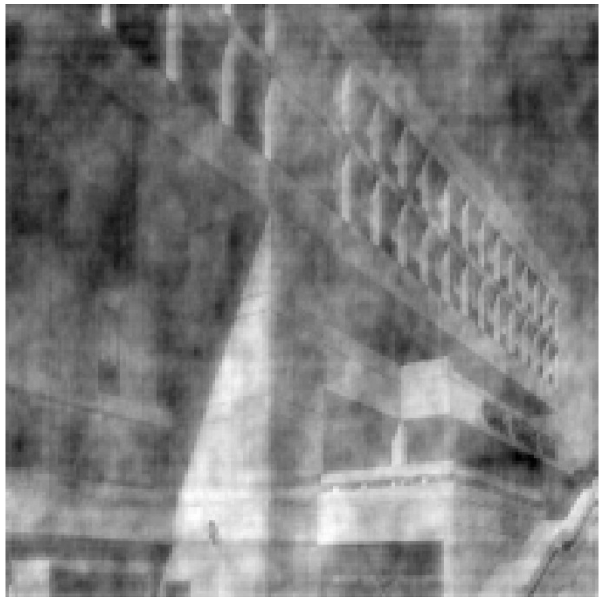
Is the phase of the image spectrum important?
Can we get away with just the magnitude?

The Importance of Phase for Visual Perception

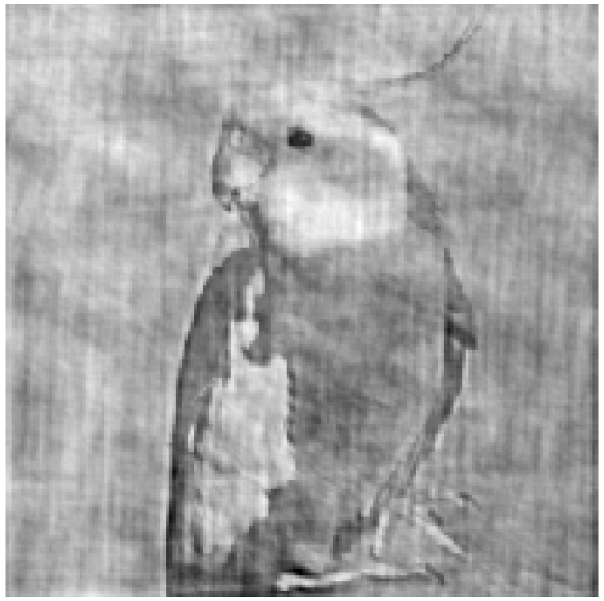
Image 1

Image 2

Magnitude(Image2), Phase(Image1)



Magnitude(Image1), Phase(Image2)



What's stored in the phase?

Outline

- Images as Functions ✓
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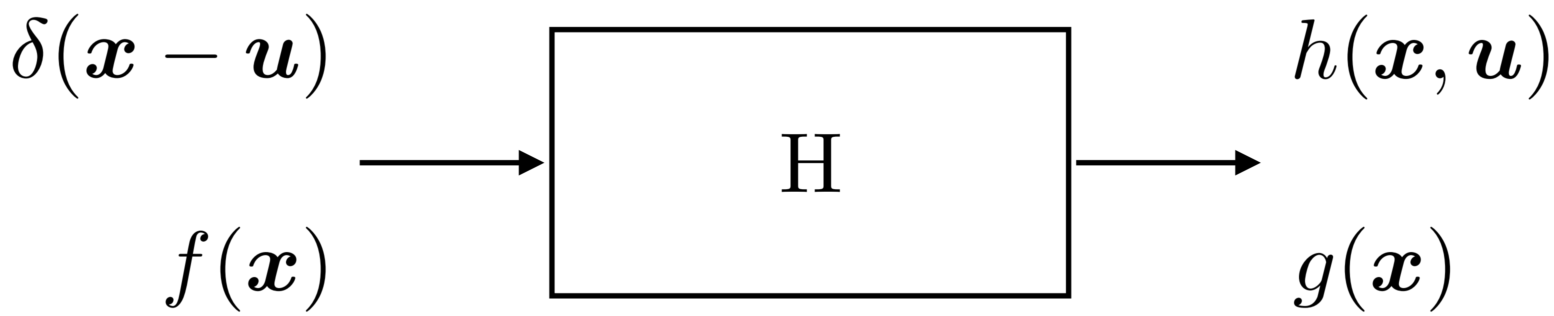
  - Two-Dimensional **Systems**
- 2D Fourier Transform ✓
  - Properties
  - Dirac Impulse, etc.
- Characterization of **LSI** Systems
  - Multidimensional **Convolution**
  - Modeling of Optical Systems
  - Examples of **Impulse Responses**

Impulse Responses of Linear Systems

Recall: It is always that $f(\boldsymbol{x}) = (\delta * f)(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{u})\delta(\boldsymbol{x}-\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}$.

Setting: Consider a linear (not necessarily shift-invariant) system H.



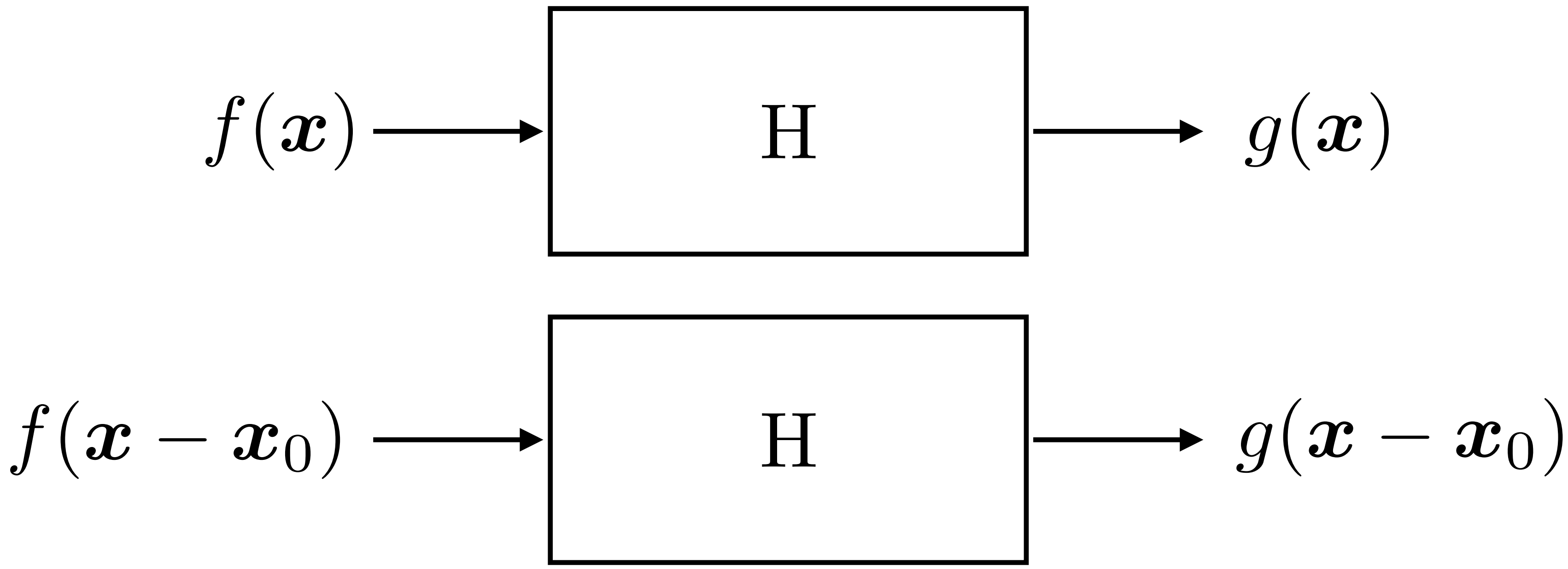
Exercise: Show that $g(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{u})h(\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}$.

$$
\begin{aligned}
g(\boldsymbol{x}) = \mathrm{H}\{f(\boldsymbol{x})\} &= \mathrm{H}\left\{\int_{\mathbb{R}^2} f(\boldsymbol{u})\delta(\boldsymbol{x}-\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\right\} \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{u})\mathrm{H}\{\delta(\boldsymbol{x}-\boldsymbol{u})\}\,\mathrm{d}\boldsymbol{u} \qquad (\mathrm{H}\text{ is linear}) \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{u})h(\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}
\end{aligned}
$$

The output of a system is the input integrated against the impulse response.

Linear, Shift-Invariant Systems (LSI)

Definition: A linear system H is **shift-invariant** if and only if shifted inputs correspond to shifted outputs.



$\forall \boldsymbol{x}_0 \in \mathbb{R}^2$

$$
\begin{aligned}
g(\boldsymbol{x}) &= \int_{\mathbb{R}^2} f(\boldsymbol{u})h(\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \\
g(\boldsymbol{x}-\boldsymbol{x}_0) &= \int_{\mathbb{R}^2} f(\boldsymbol{u})h(\boldsymbol{x}-\boldsymbol{x}_0,\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \\
g(\boldsymbol{x}-\boldsymbol{x}_0) &= \int_{\mathbb{R}^2} f(\boldsymbol{u}-\boldsymbol{x}_0)h(\boldsymbol{x},\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \\
&= \int_{\mathbb{R}^2} f(\boldsymbol{u})h(\boldsymbol{x},\boldsymbol{u}+\boldsymbol{x}_0)\,\mathrm{d}\boldsymbol{u}
\end{aligned}
$$

$$\implies h(\boldsymbol{x}-\boldsymbol{x}_0,\boldsymbol{u}) = h(\boldsymbol{x},\boldsymbol{u}+\boldsymbol{x}_0)$$

$\forall \boldsymbol{x},\boldsymbol{x}_0,\boldsymbol{u} \in \mathbb{R}^2$

In particular, $h(\boldsymbol{x}-\boldsymbol{x}_0,\mathbf{0}) = h(\boldsymbol{x},\boldsymbol{x}_0)$

(set $\boldsymbol{u}=\mathbf{0}$)

Linear, Shift-Invariant Systems (LSI)

Definition: A linear system H is **shift-invariant** if and only if shifted inputs correspond to shifted outputs.

$$f(\boldsymbol{x}) \longrightarrow \boxed{\quad \mathrm{H} \quad} \longrightarrow g(\boldsymbol{x})$$

$$g(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{u}) h(\boldsymbol{x}, \boldsymbol{u}) \, \mathrm{d}\boldsymbol{u}$$

LSI systems are realized by convolutions!

$h(\boldsymbol{x} - \boldsymbol{x}_0, \mathbf{0}) = h(\boldsymbol{x}, \boldsymbol{x}_0)$ for any $\boldsymbol{x}, \boldsymbol{x}_0 \in \mathbb{R}^2$

Define $h_{\mathrm{LSI}}(\boldsymbol{x}) = h(\boldsymbol{x}, \mathbf{0})$

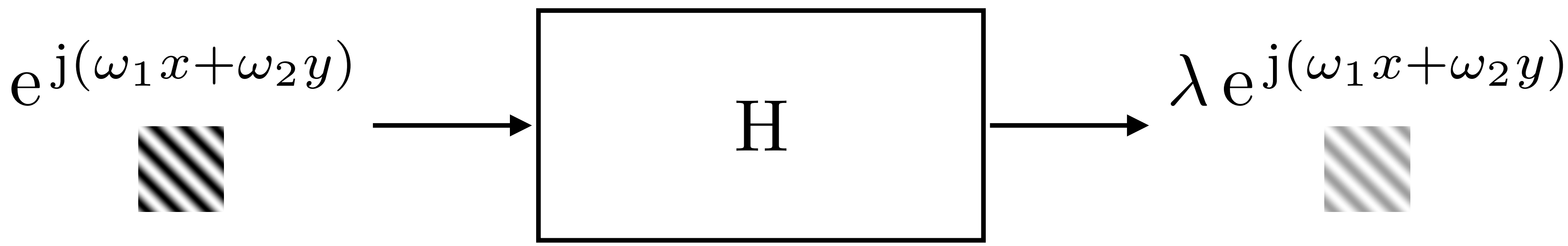
$$\implies h(\boldsymbol{x}, \boldsymbol{u}) = h(\boldsymbol{x} - \boldsymbol{u}, \mathbf{0}) = h_{\mathrm{LSI}}(\boldsymbol{x} - \boldsymbol{u})$$

We will just write h for h_{LSI}

$$g(\boldsymbol{x}) = \mathrm{H}\{f(\boldsymbol{x})\} = \int_{\mathbb{R}^2} f(\boldsymbol{u}) h_{\mathrm{LSI}}(\boldsymbol{x} - \boldsymbol{u}) \, \mathrm{d}\boldsymbol{u} = (f * h_{\mathrm{LSI}})(\boldsymbol{x})$$

Linear, Shift-Invariant Systems (LSI)

Theorem: Complex exponentials $\mathrm{e}^{\mathrm{j}(\omega_1 x+\omega_2 y)}$ are *eigenfunctions* of the LSI system H with *eigenvalue* $\lambda = \lambda(\omega_1, \omega_2) = \hat{h}(\omega_1, \omega_2)$.



Proof:
$$\begin{aligned}\mathrm{H}\{\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}}\} &= \int_{\mathbb{R}^2} h(\boldsymbol{u})\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}(\boldsymbol{x}-\boldsymbol{u})}\,\mathrm{d}\boldsymbol{u}\\ &= \mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}}\int_{\mathbb{R}^2} h(\boldsymbol{u})\mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{u}}\,\mathrm{d}\boldsymbol{u}\\ &= \hat{h}(\boldsymbol{\omega})\,\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}}\end{aligned}$$

2D Convolutions

- 2D Convolution integral

$$(f * h)(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u, v) h(x - u, y - v) \,\mathrm{d}u \,\mathrm{d}v$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u, v) f(x - u, y - v) \,\mathrm{d}u \,\mathrm{d}v = (h * f)(x, y)$$

- Convolution theorem

$$(f * h)(\boldsymbol{x}) \overset{\mathcal{F}}{\longleftrightarrow} \hat{f}(\boldsymbol{\omega}) \hat{h}(\boldsymbol{\omega})$$

The Fourier transform converts convolutions into multiplications!

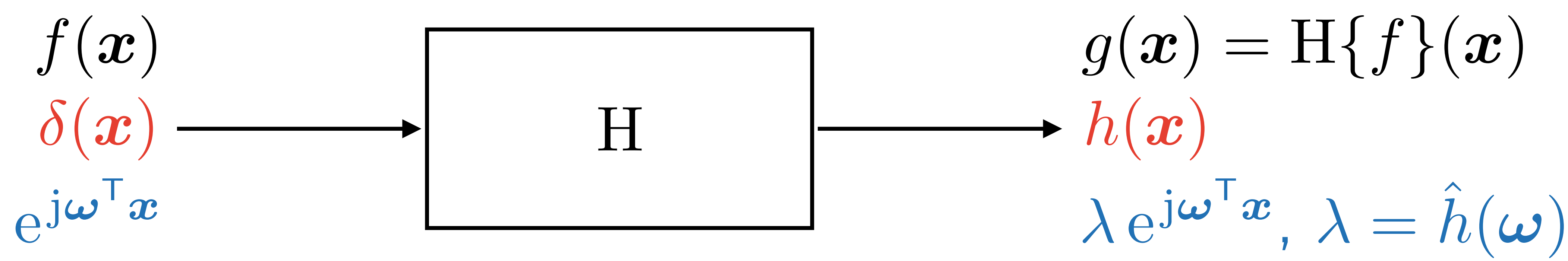
Proof of the Convolution Theorem

$$g(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{u})h(\boldsymbol{x}-\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}$$

$$\Big\updownarrow \mathcal{F}$$

$$\begin{aligned}
\hat{g}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} f(\boldsymbol{u})h(\boldsymbol{x}-\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \right) \mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} f(\boldsymbol{u})h(\boldsymbol{v})\mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}(\boldsymbol{u}+\boldsymbol{v})}\,\mathrm{d}\boldsymbol{u} \right) \mathrm{d}\boldsymbol{v} && \text{(change of variables } \boldsymbol{v} = \boldsymbol{x}-\boldsymbol{u}\text{)} \\
&= \left(\int_{\mathbb{R}^2} f(\boldsymbol{u})\mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{u}}\,\mathrm{d}\boldsymbol{u} \right) \left(\int_{\mathbb{R}^2} h(\boldsymbol{v})\mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{v}}\,\mathrm{d}\boldsymbol{v} \right) && (\mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}(\boldsymbol{u}+\boldsymbol{v})} = \mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{u}}\mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{v}}) \\
&= \hat{f}(\boldsymbol{\omega})\hat{h}(\boldsymbol{\omega})
\end{aligned}$$

Frequency Responses of LSI Systems



Linear and Shift-Invariant (LSI)

What is the frequency response of this system?

- The frequency response is $\hat{h}(\boldsymbol{\omega})$

Controls notation: $H(\mathrm{j}\boldsymbol{\omega}) = \hat{h}(\boldsymbol{\omega})$

$H(\boldsymbol{s})$, $\boldsymbol{s} \in \mathbb{C}^2$, is the **transfer function**

$$H(\boldsymbol{s}) = \int_{\mathbb{R}^2} h(\boldsymbol{x}) \mathrm{e}^{-\boldsymbol{s}^\mathsf{T}\boldsymbol{x}} \, \mathrm{d}\boldsymbol{x} \qquad \text{(2D two-sided Laplace transform)}$$

How do we identify the frequency response of a system?

Identification of the Frequency Response

$$
\begin{array}{c} f(\boldsymbol{x}) \\ \delta(\boldsymbol{x}) \\ \mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}} \end{array}
\longrightarrow \boxed{\mathrm{H}} \longrightarrow
\begin{array}{l} g(\boldsymbol{x}) = \mathrm{H}\{f\}(\boldsymbol{x}) \\ h(\boldsymbol{x}) \\ \lambda\, \mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}}, \ \lambda = \hat{h}(\boldsymbol{\omega}) \end{array}
$$

Linear and Shift-Invariant (LSI)

- Method 1: Eigenfunction property

  Excite the system with
  a pure frequency

$$\hat{h}(\boldsymbol{\omega}) = \lambda = \frac{\mathrm{H}\{\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}}\}}{\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}}}$$

- Method 2: From the impulse response

  Excite the system with
  an impulse

$$\hat{h}(\boldsymbol{\omega}) = \widehat{\mathrm{H}\{\delta\}}(\boldsymbol{\omega})$$

- Method 3: From an arbitrary input and output

  Excite the system with
  an arbitrary input

$$g(\boldsymbol{x}) = (f * h)(\boldsymbol{x}) \implies \hat{g}(\boldsymbol{\omega}) = \hat{f}(\boldsymbol{\omega})\hat{h}(\boldsymbol{\omega})$$

$$\implies \hat{h}(\boldsymbol{\omega}) = \frac{\hat{g}(\boldsymbol{\omega})}{\hat{f}(\boldsymbol{\omega})}$$

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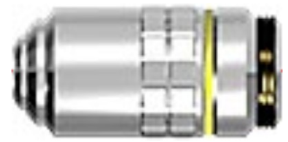
Modeling of Optical Systems

$f(\boldsymbol{x}) \longrightarrow$  $\longrightarrow g(\boldsymbol{x}) = (h * f)(\boldsymbol{x})$

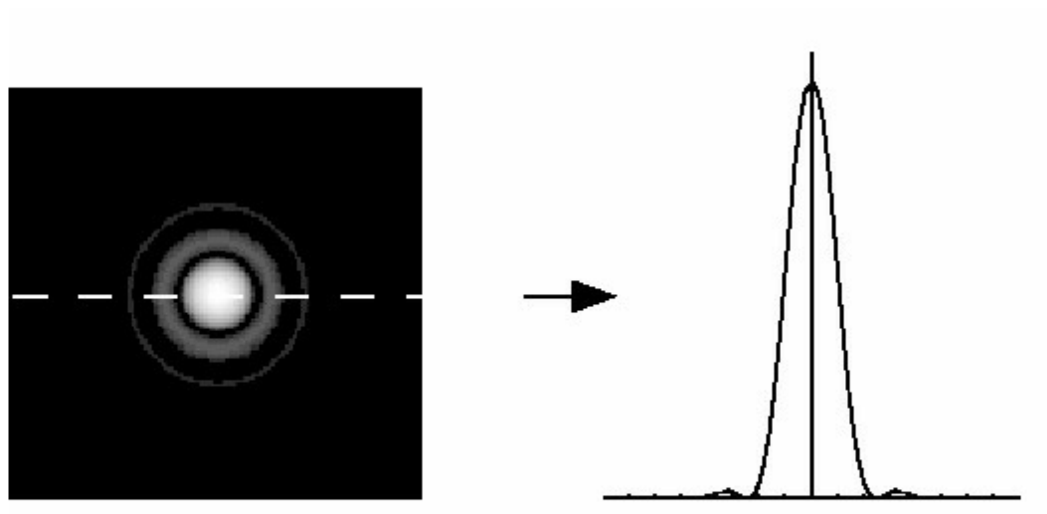
$h(\boldsymbol{x}) = h(x, y)$: Point Spread Function (PSF)

Diffraction-limited optics $=$ LSI system

microscopes,
telescopes,
cameras, etc.

- Aberration-free point spread function

  "ideal"



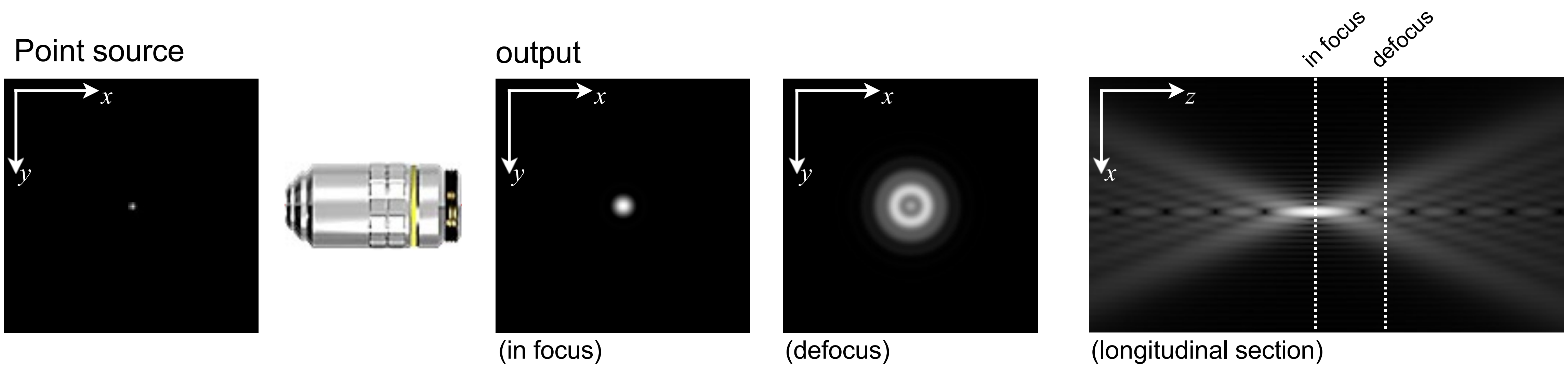
Airy Disk　　Radial Profile

Modeling of Optical Systems

$f(\boldsymbol{x})$ ⟶  ⟶ $g(\boldsymbol{x}) = (h * f)(\boldsymbol{x})$

$h(\boldsymbol{x}) = h(x, y)$: Point Spread Function (PSF)

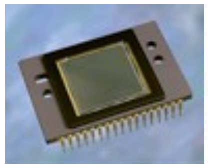
- Effect of misfocus



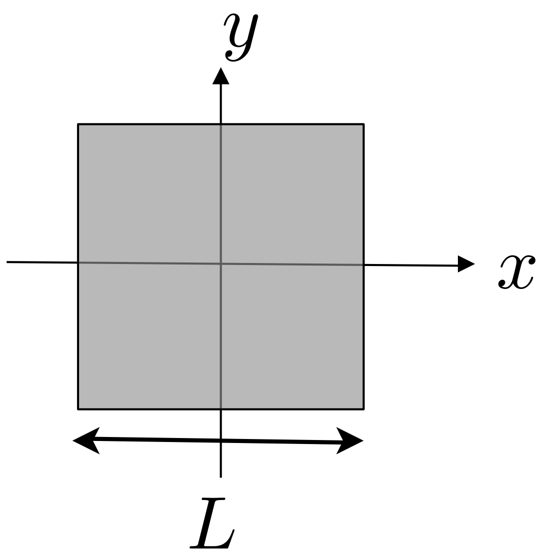
Need to be mindful of the focal length of your optical system

More Examples of System Responses

- CCD (= digital) camera



Impulse response = sampling aperature = photosite = pixel integration area



$$h(x,y) = \frac{1}{L^2}\mathrm{rect}\left(\frac{x}{L}\right)\,\mathrm{rect}\left(\frac{y}{L}\right)$$

$$\Big\updownarrow \mathcal{F}$$

$$\hat{h}(\omega_1,\omega_2) = \mathrm{sinc}\left(\frac{L\omega_1}{2\pi}\right)\,\mathrm{sinc}\left(\frac{L\omega_2}{2\pi}\right)$$

The resolution of CCD cameras is $L \times L$

This system takes **local averages** over $L \times L$ squares

More Examples of System Responses

- Motion blur

time-varying position

Hypothesis: "system" comes from the motion of the camera: $\boldsymbol{x}_0(t)$

$$g(\boldsymbol{x}) = \frac{1}{T}\int_0^T f(\boldsymbol{x} - \boldsymbol{x}_0(t))\,\mathrm{d}t$$

Is this system shift-invariant?

Shift-invariant, but not time-invariant

What is the impulse response?

$$h(\boldsymbol{x}) = \frac{1}{T}\int_0^T \delta(\boldsymbol{x} - \boldsymbol{x}_0(t))\,\mathrm{d}t$$

$$\Big\updownarrow \mathcal{F}$$

What is the frequency response?

$$\hat{h}(\boldsymbol{\omega}) = \frac{1}{T}\int_0^T \mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}_0(t)}\,\mathrm{d}t$$

Motion Blur Example

- Uniform motion $\boldsymbol{x}_0(t) = (t/T, 0)$

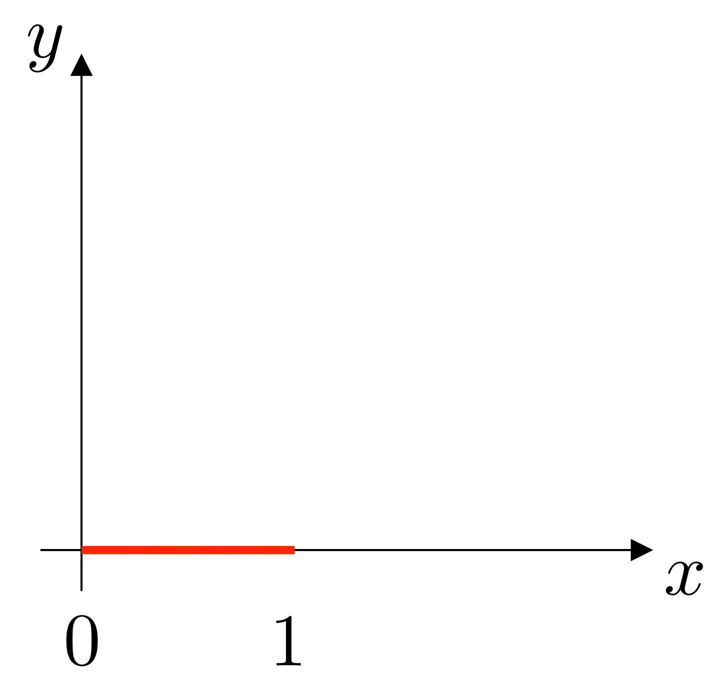
$$h(x,y) = \frac{1}{T}\int_0^T \delta\left(x - \frac{t}{T}, y\right)\,\mathrm{d}t$$

$$= \frac{1}{T}\int_0^T \delta\left(x - \frac{t}{T}\right)\delta(y)\,\mathrm{d}t$$

$$= \delta(y)\,\frac{1}{T}\int_0^T \delta\left(x - \frac{t}{T}\right)\,\mathrm{d}t$$

$$= \delta(y)\,\frac{1}{T}\int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{t}{T} - \frac{1}{2}\right)\delta\left(x - \frac{t}{T}\right)\,\mathrm{d}t$$

$$= \mathrm{rect}\left(x - \frac{1}{2}\right)\delta(y) \overset{\mathcal{F}}{\longleftrightarrow} \hat{h}(\omega_1, \omega_2) = \mathrm{e}^{-\mathrm{j}\omega_1/2}\,\mathrm{sinc}\left(\frac{\omega_1}{2\pi}\right)$$



The impulse response is a “windowed” ideal line

Motion Blur Example

Can we use this to understand photographing a moving car?

Change the frame of reference to the car

This is how motion blurs are implemented in Photoshop, GIMP, etc.

Summary

- Analog images are modeled as functions $f(x, y)$ of the two spatial variables x and y.
- These functions are assumed to have finite energy: $f \in L^2(\mathbb{R}^2)$. It is convenient to view them as points in a vector space with an inner product.
- An image-processing operator (or system) is a mapping $\mathrm{H} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$.
- The complex exponentials $\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{x}}$ are the eigenfunctions of LSI systems. They are $(2\pi/\|\boldsymbol{\omega}\|_2)$-periodic plane waves that propagate in the direction $\boldsymbol{\omega}$.
- The 2D Fourier transform of an image reveals its spatial frequency content. The Fourier phase contains the most relevant perceptual information (contours).
- The 2D Fourier transform is very similar to the 1D one and therefore inherits essentially the same properties.
- The 2D Fourier transform of a separable image $f(x, y) = f_1(x)f_2(y)$ is determined using 1D transforms only.
- LSI systems are realized by convolutions.
- Analog LSI systems are completely characterized by their impulse response (point-spread function or sampling aperature) $h(x, y) = \mathrm{H}\{\delta\}(x, y)$, or, equivalently, by their frequency response $\hat{h}(\omega_1, \omega_2)$.