ECE 172A: Introduction to Image Processing
Sampling and Acquisition of Images: Part I

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Outline

- Sampling Theory
  - Review 1D Sampling Theory
  - Sampling in Two Dimensions
- Acquisition Systems
  - Real Acquisition Systems
  - Aliasing Problems
- Image Quantization
  - Uniform Quantizer
  - Minimum-Error (Lloyd-Max) Quantizer
  - Grayscale vs. Spatial Resolution Tradeoff

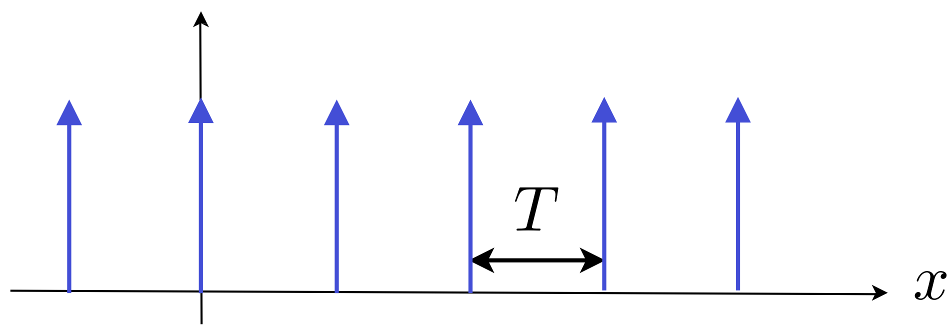
Sampling Theory

- Review 1D Sampling Theory
- Sampling in Fourier Domain
- Sampling and Aliasing
- Shannon's Sampling Theorem
- Sampling in 2D

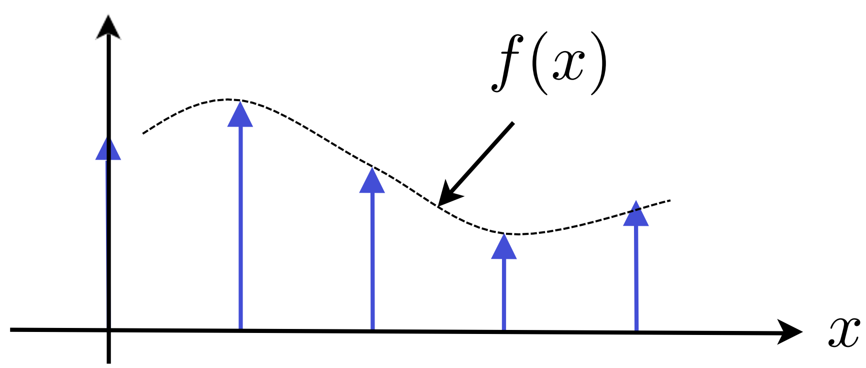
Review of 1D Sampling Theory

- Ideal sampling = multiplication with a **Dirac comb**

$$Ш_T(x) = \sum_{k \in \mathbb{Z}} \delta(x - kT)$$



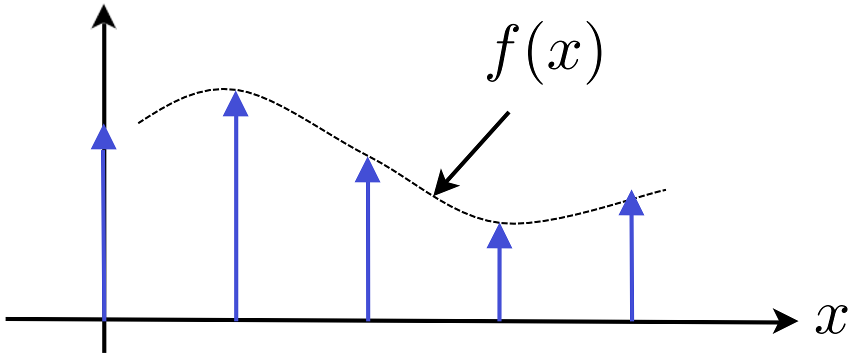
$$\begin{aligned} f_T(x) &= f(x) Ш_T(x) \\ &= f(x) \sum_{k \in \mathbb{Z}} \delta(x - kT) \\ &= \sum_{k \in \mathbb{Z}} f(x) \delta(x - kT) \\ &= \sum_{k \in \mathbb{Z}} f(kT) \delta(x - kT) \end{aligned}$$



Review of 1D Sampling Theory (cont'd)

- Ideal sampling = multiplication with a **Dirac comb**

$$f_T(x) = \sum_{k \in \mathbb{Z}} f(kT)\delta(x - kT)$$



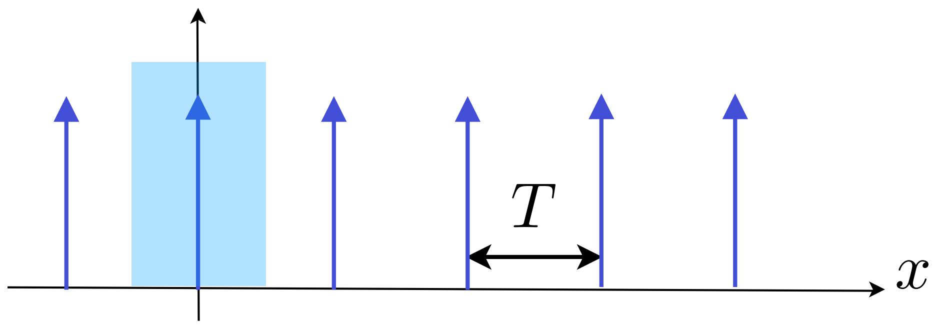
- Sampling periodizes the Fourier domain

$$\begin{aligned}\hat{f}_T(\omega) = \int_{-\infty}^{\infty} f_T(x)\mathrm{e}^{-\mathrm{j}\omega x}\,\mathrm{d}x &= \int_{-\infty}^{\infty} \sum_{k \in \mathbb{Z}} f(kT)\delta(x - kT)\mathrm{e}^{-\mathrm{j}\omega x}\,\mathrm{d}x \\ &= \sum_{k \in \mathbb{Z}} f(kT) \int_{-\infty}^{\infty} \delta(x - kT)\mathrm{e}^{-\mathrm{j}\omega x}\,\mathrm{d}x \\ &= \sum_{k \in \mathbb{Z}} f(kT)\mathrm{e}^{-\mathrm{j}\omega kT}\end{aligned}$$

This is a $(2\pi/T)$-periodic function in ω

Dirac Comb Fourier Transform

$$\text{Ш}_T(x) = \sum_{k\in\mathbb{Z}} \delta(x - kT) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad ???$$



- T-periodic functions can be specified by their **Fourier series**

$$f_T(x) = \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{\mathrm{j}n\omega_0 x} \quad \text{with} \quad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(x)\mathrm{e}^{-\mathrm{j}n\omega_0 x}\,\mathrm{d}x, \quad \omega_0 = \frac{2\pi}{T}$$

$$\text{Ш}_T(x) = \sum_{n\in\mathbb{Z}} \left(\frac{1}{T}\int_{-T/2}^{T/2} \text{Ш}_T(x)\mathrm{e}^{-\mathrm{j}n\omega_0 x}\,\mathrm{d}x\right)\mathrm{e}^{\mathrm{j}n\omega_0 x}$$

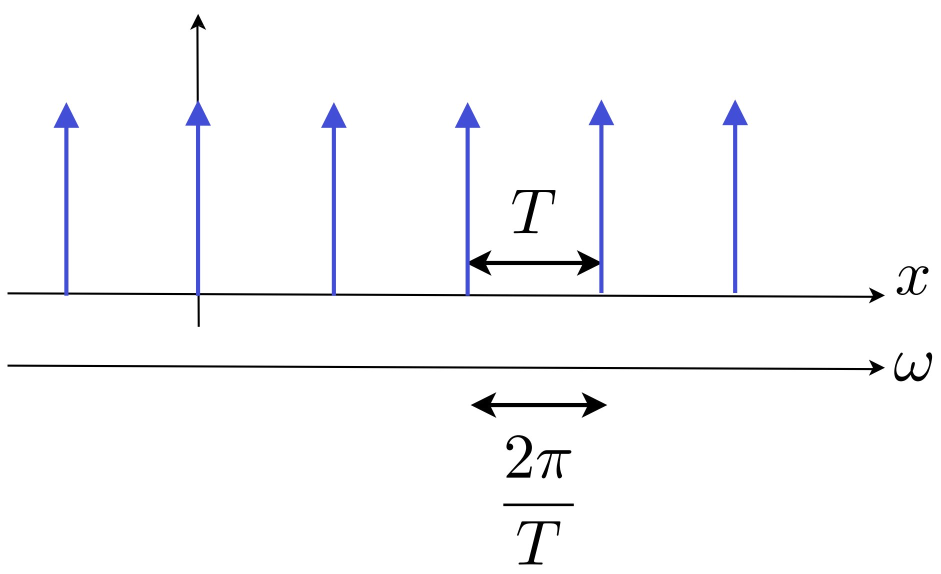
$$= \sum_{n\in\mathbb{Z}} \left(\frac{1}{T}\int_{-T/2}^{T/2} \delta(x)\mathrm{e}^{-\mathrm{j}n\omega_0 x}\,\mathrm{d}x\right)\mathrm{e}^{\mathrm{j}n\omega_0 x}$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{T}\mathrm{e}^{\mathrm{j}n\omega_0 x} \quad \text{with} \quad \omega_0 = \frac{2\pi}{T}$$

Dirac Comb Fourier Transform (cont'd)

$$Ш_T(x) = \sum_{k\in\mathbb{Z}} \delta(x - kT) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad ???$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{T} e^{jn\omega_0 x}$$



$$e^{jn\omega_0 x} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad 2\pi\delta(\omega - n\omega_0) \quad \text{(by duality)}$$

$$Ш_T(x) = \sum_{n\in\mathbb{Z}} \frac{1}{T} e^{jn\omega_0 x} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \sum_{n\in\mathbb{Z}} \frac{1}{T} 2\pi\delta(\omega - n\omega_0)$$

$$= \frac{2\pi}{T} \sum_{n\in\mathbb{Z}} \delta\left(\omega - \frac{2\pi}{T} n\right)$$

The Fourier transform of a Dirac comb is a Dirac comb

Sampling in the Fourier Domain

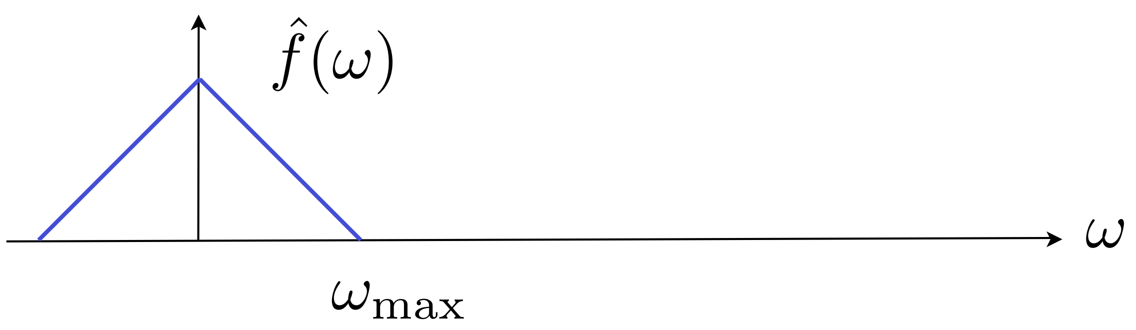
- Sampling formula

$$f_T(x) = f(x) \cdot \sum_{k\in\mathbb{Z}} \delta(x - kT) \quad \xleftrightarrow{\mathcal{F}} \quad \frac{1}{2\pi}\left(\hat{f}(\omega) * \frac{2\pi}{T}\sum_{n\in\mathbb{Z}} \delta\left(\omega - \frac{2\pi n}{T}\right)\right)$$

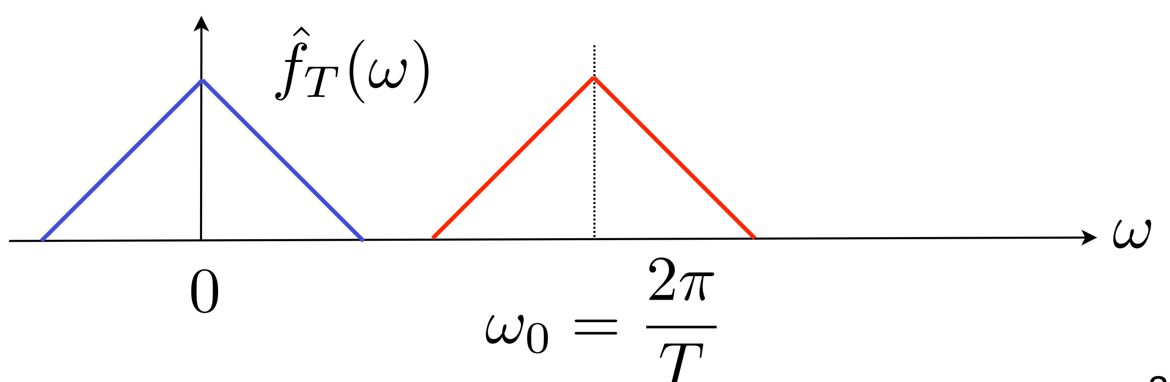
$$\hat{f}_T(\omega) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \hat{f}\left(\omega - \frac{2\pi n}{T}\right) = \sum_{k\in\mathbb{Z}} f(kT)\mathrm{e}^{-\mathrm{j}\omega kT}$$

- Sampling **periodizes** the spectrum

$\hat{f}(\omega)$: Fourier transform of $f(x)$
(continuous-domain function)



$\hat{f}_T(\omega)$: $(2\pi/T)$-periodization of $\hat{f}(x)$
(continuous-domain function)



Sampling in the Fourier Domain

$$\hat{f}_T(\omega) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \hat{f}\left(\omega - \frac{2\pi n}{T}\right) = \sum_{k\in\mathbb{Z}} f(kT)\mathrm{e}^{-\mathrm{j}\omega kT}$$

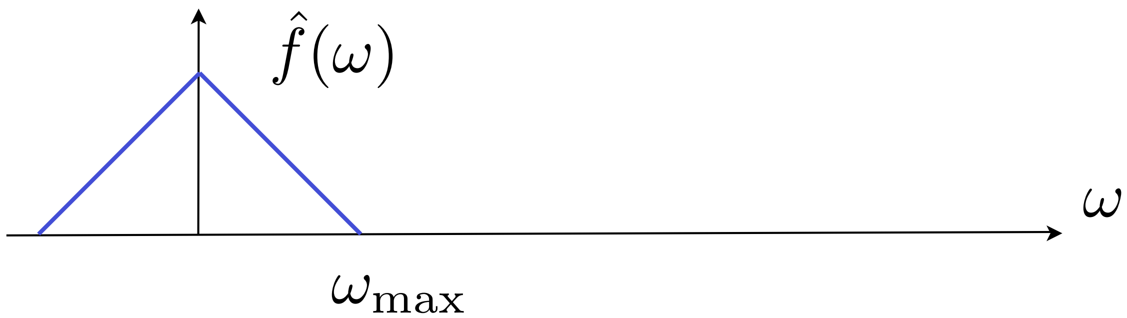
- Normalized sampling step ($T = 1$)

$$\hat{f}_{T=1}(\omega) = \sum_{n\in\mathbb{Z}} \hat{f}(\omega - 2\pi n) = \sum_{k\in\mathbb{Z}} f(k)\mathrm{e}^{-\mathrm{j}\omega k}$$

Poisson summation formula

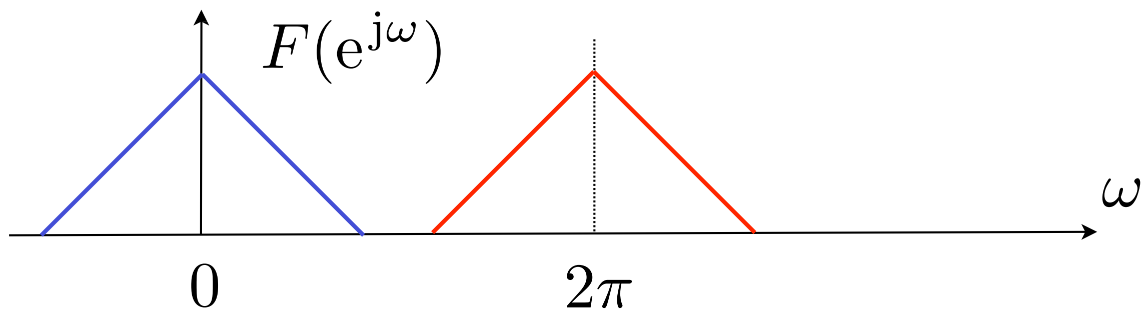
- Define the discrete sequence $f[k] = f(k)$, $k \in \mathbb{Z}$

$\hat{f}(\omega)$: Fourier transform of $f(x)$ (continuous-domain function)



$\hat{f}_{T=1}(\omega)$: Discrete-~~time~~ space Fourier transform of $f[k]$

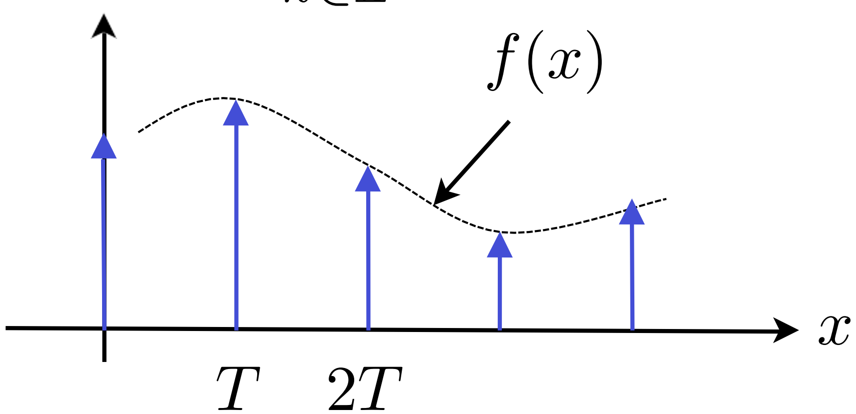
$$F(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{k\in\mathbb{Z}} f[k]\mathrm{e}^{-\mathrm{j}\omega k}$$



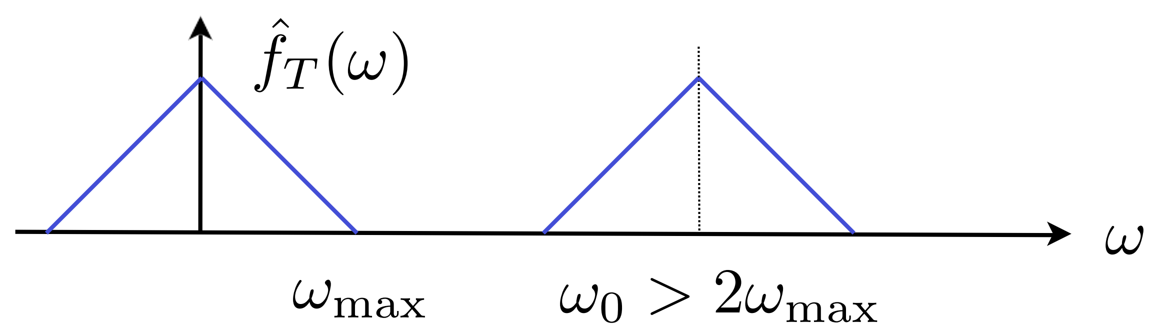
Sampling and Aliasing

- Sampled signal

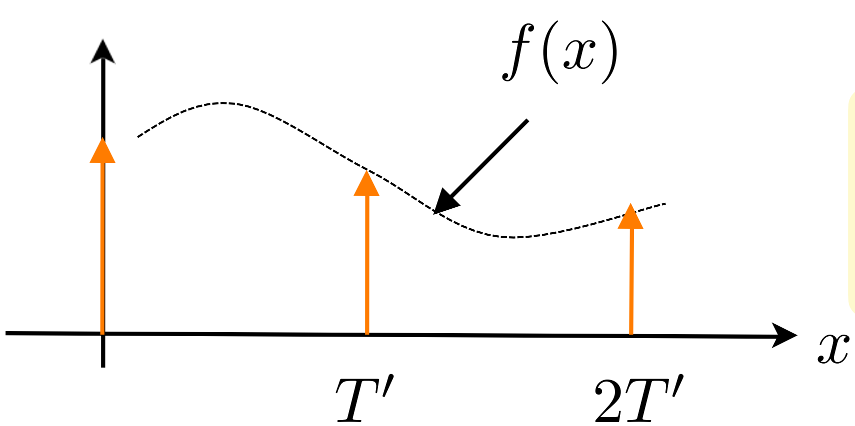
$$f_T(x) = \sum_{k \in \mathbb{Z}} f(kT)\delta(x - kT) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \hat{f}_T(\omega) = \frac{1}{T} \sum_{n \in \mathbb{Z}} \hat{f}\left(\omega - \frac{2\pi n}{T}\right)$$



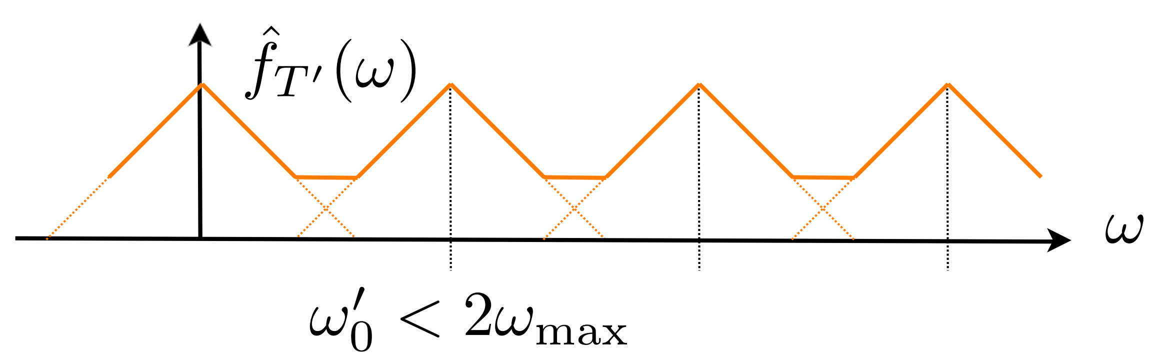
$$\omega_0 = \frac{2\pi}{T}$$



"undersampling"



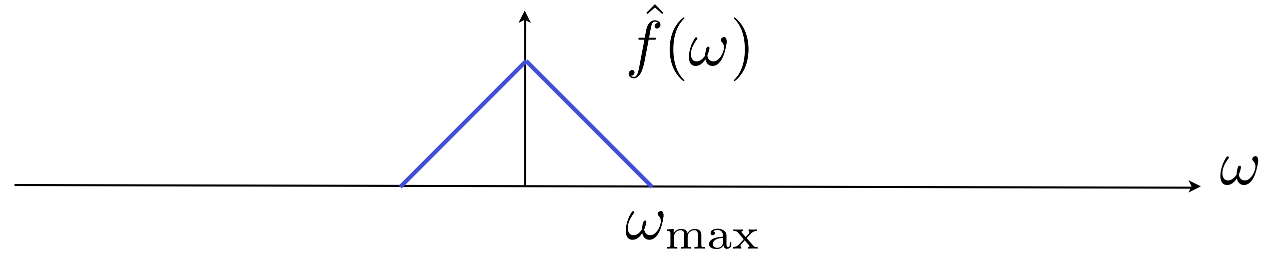
$$\omega_0' = \frac{2\pi}{T'}$$



Undersampling causes **aliasing** which destroys the spectrum

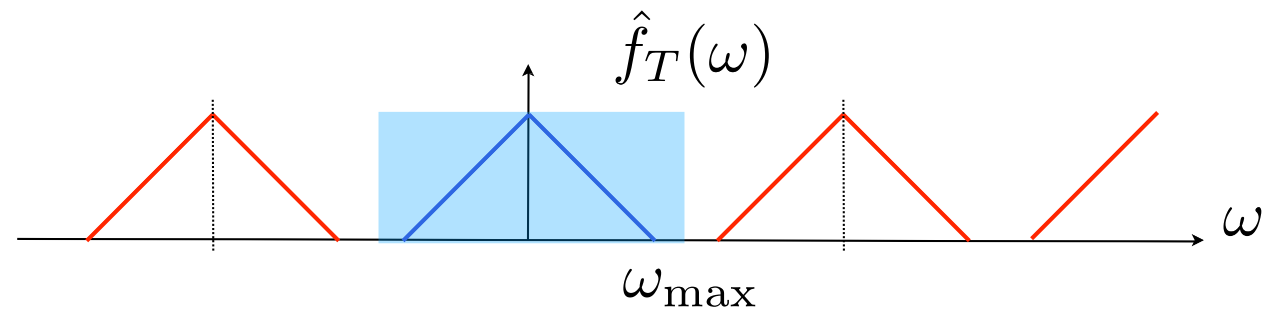
Sampling and Reconstruction

- Continuous-domain input signal $f(x)$



- Sampling

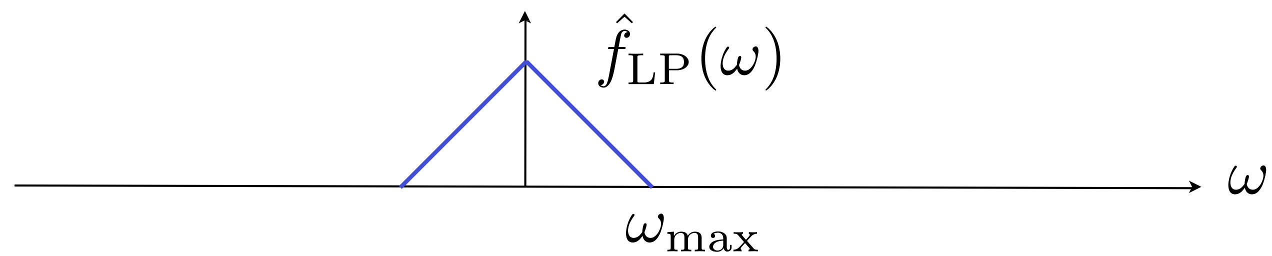
$$\hat{f}_T(\omega) = \frac{1}{T} \sum_{n \in \mathbb{Z}} \hat{f}\left(\omega - \frac{2\pi n}{T}\right)$$



- Ideal low-pass filtering

$$\hat{f}_{\mathrm{LP}}(\omega) = \hat{f}_T(\omega) T \operatorname{rect}\left(\frac{\omega T}{2\pi}\right)$$

$$= \begin{cases} \sum_{n \in \mathbb{Z}} \hat{f}\left(\omega - \frac{2\pi n}{T}\right), & |\omega| < \frac{\pi}{T} \\ 0, & \text{else} \end{cases}$$



When does $\hat{f}_{\mathrm{LP}} = \hat{f}$?

When do we have perfect recovery?

Nyquist–Shannon Sampling Theorem

(Whittaker–Nyquist–Kotelnikov–Shannon Sampling Theorem)

- Ideal Reconstruction Process

Exercise:

$$\mathrm{sinc}\left(\frac{x}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} ???$$

$\delta(x)$

$$\sum_{k\in\mathbb{Z}} f(kT)\,\delta(x-kT)$$

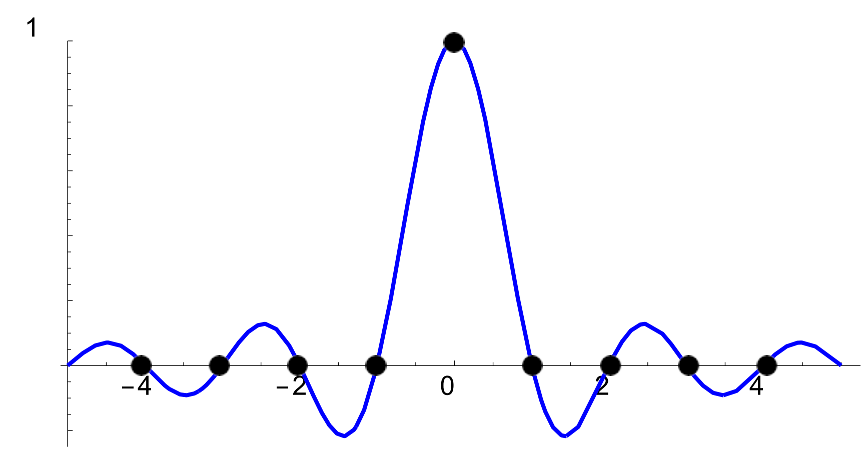
→ Ideal low-pass filter →

$$f_{\mathrm{LP}}(x) = \sum_{k\in\mathbb{Z}} f(kT)\,\mathrm{sinc}\left(\frac{x-kT}{T}\right)$$

$$\mathrm{sinc}(x) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$

$$f\left(\frac{x}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} T\,\hat{f}(T\omega)$$

$$\mathrm{sinc}\left(\frac{x}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} T\,\mathrm{rect}\left(\frac{T\omega}{2\pi}\right)$$



Nyquist–Shannon Sampling Theorem

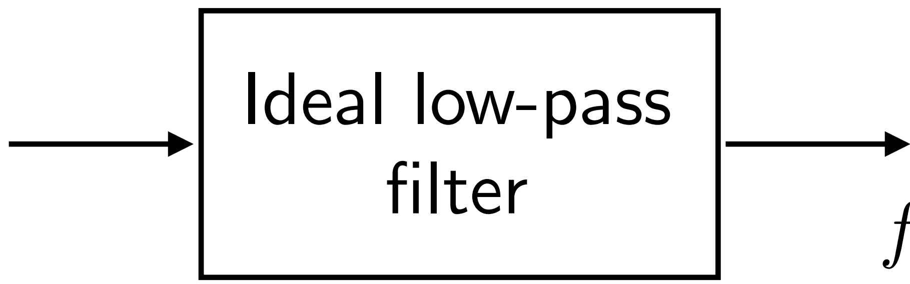
(Whittaker–Nyquist–Kotelnikov–Shannon Sampling Theorem)

- Ideal Reconstruction Process

$$\mathrm{sinc}\left(\frac{x}{T}\right) \xleftrightarrow{\mathcal{F}} T\,\mathrm{rect}\left(\frac{T\omega}{2\pi}\right)$$

$\delta(x)$

$$\sum_{k\in\mathbb{Z}} f(kT)\,\delta(x-kT)$$



$$f_{\mathrm{LP}}(x) = \sum_{k\in\mathbb{Z}} f(kT)\,\mathrm{sinc}\left(\frac{x-kT}{T}\right)$$

Sampling Theorem

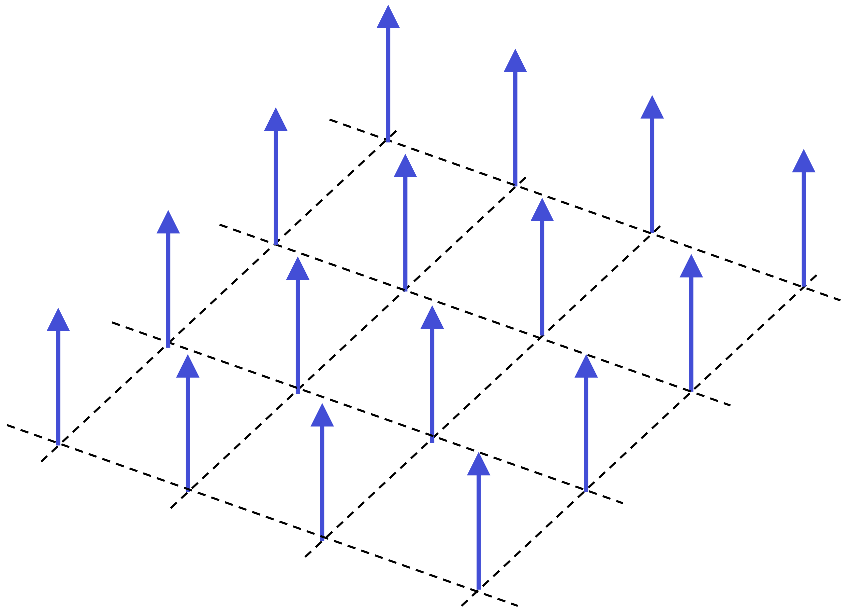
A function $f(x)$ that is bandlimited to $\omega_{\max}$ can be reconstructed exactly from its equidistant samples provided that the sampling step $T < \pi/\omega_{\max}$. Specifically,

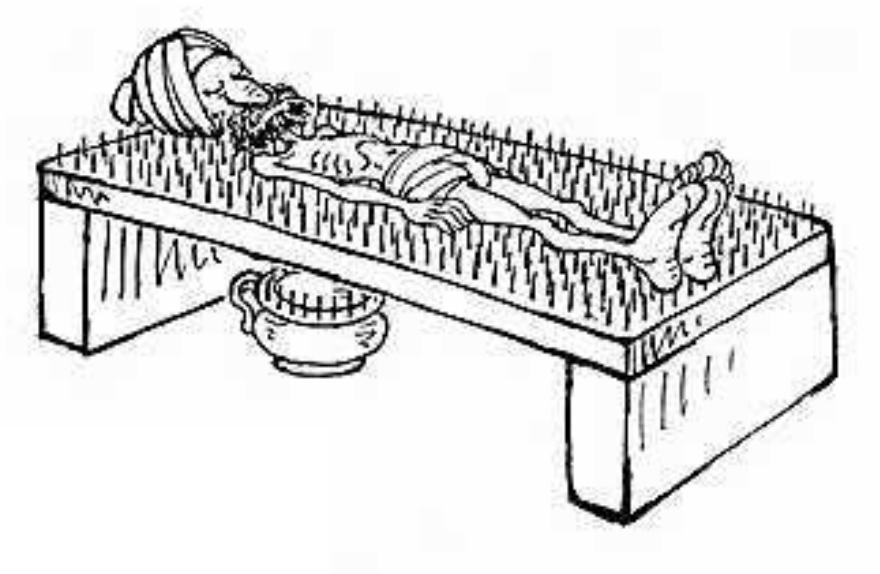
$$f(x) = \sum_{k\in\mathbb{Z}} f(kT)\mathrm{sinc}\left(\frac{x-kT}{T}\right)$$

where $\mathrm{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x} \xleftrightarrow{\mathcal{F}} \mathrm{rect}\left(\dfrac{\omega}{2\pi}\right)$.

Sampling in 2D

How do we sample in 2D?





- 2D Dirac comb

$$\sum_{(k,l)\in\mathbb{Z}^2} \delta(x - kT_1, y - lT_2) \quad \xleftrightarrow{\mathcal{F}} \quad \frac{(2\pi)^2}{T_1 T_2} \sum_{(m,n)\in\mathbb{Z}^2} \delta\left(\omega_1 - \frac{2\pi m}{T_1}, \omega_2 - \frac{2\pi n}{T_2}\right)$$

- 2D sampling formula

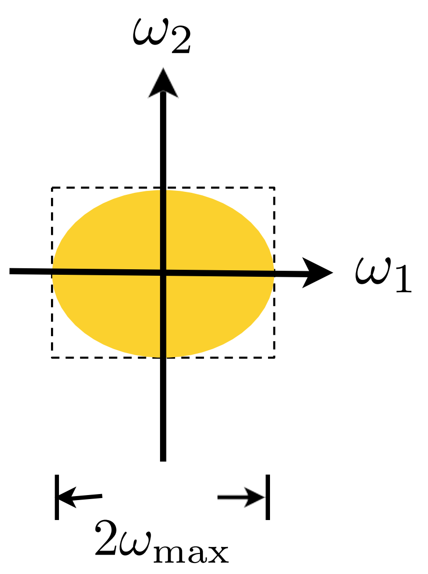
$$f_{T_1,T_2}(x, y) = f(x, y) \sum_{(k,l)\in\mathbb{Z}^2} \delta(x - kT_1, y - lT_2)$$

$$\xleftrightarrow{\mathcal{F}} \qquad \hat{f}_{T_1,T_2}(\omega_1, \omega_2) = \frac{1}{T_1 T_2} \sum_{(m,n)\in\mathbb{Z}^2} \hat{f}\left(\omega_1 - \frac{2\pi m}{T_1}, \omega_2 - \frac{2\pi n}{T_2}\right)$$

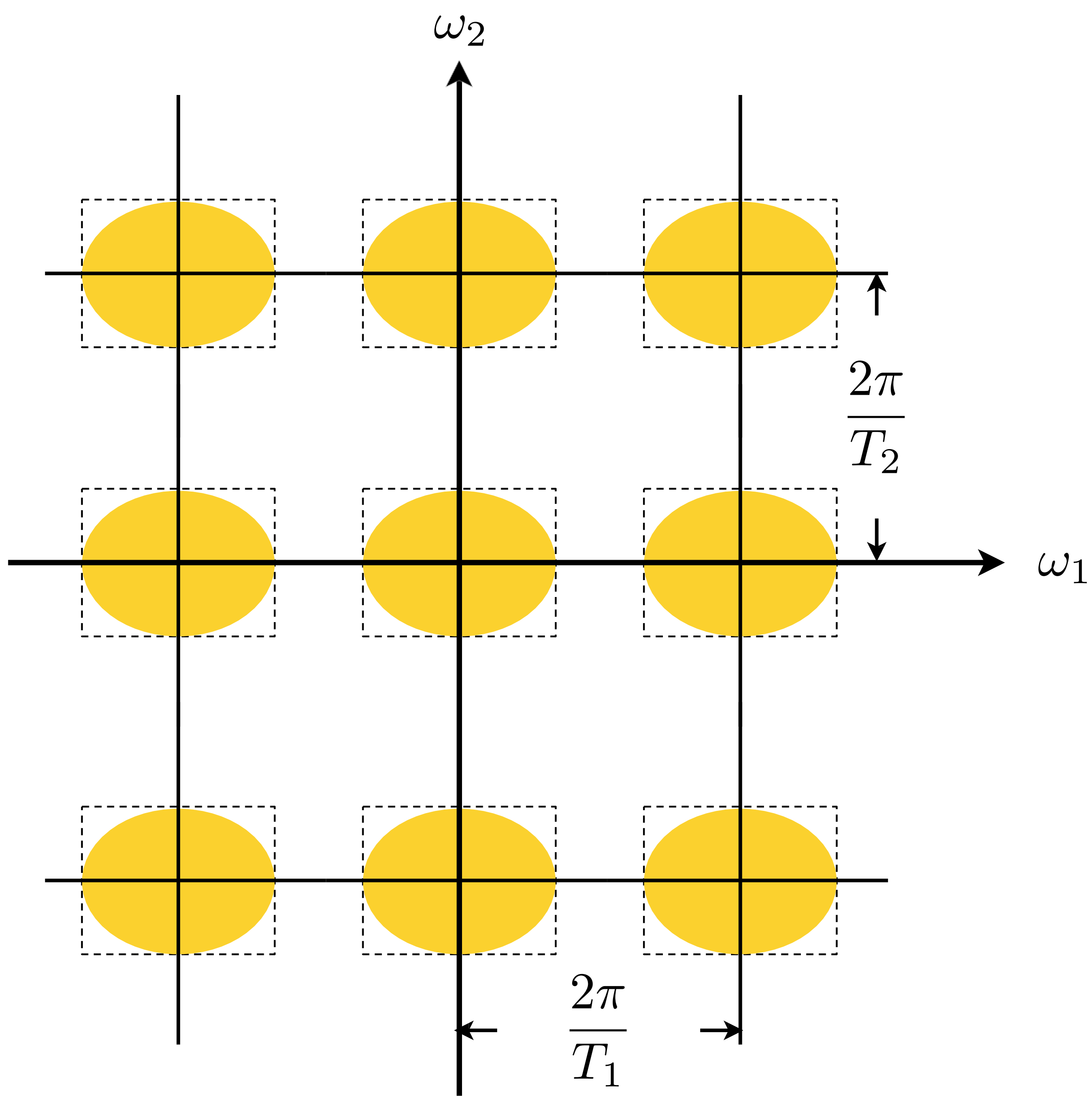
Sampling and Spectral Repitition

$$\hat{f}_{T_1,T_2}(\omega_1,\omega_2) = \frac{1}{T_1 T_2} \sum_{(m,n)\in\mathbb{Z}^2} \hat{f}\left(\omega_1 - \frac{2\pi m}{T_1}, \omega_2 - \frac{2\pi n}{T_2}\right)$$

$\hat{f}(\boldsymbol{\omega})$



spectral support



Sampling in 2D

(For notational simplicity, let $T_1 = T_2$, i.e., the sampling grid is **regular**)

- Sampling function in two dimensions

$$\text{Ш}_T(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^2} \delta(\boldsymbol{x} - \boldsymbol{k}T) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \widehat{\text{Ш}}_T(\boldsymbol{\omega}) = \frac{(2\pi)^2}{T^2} \sum_{\boldsymbol{n} \in \mathbb{Z}^2} \delta\left(\boldsymbol{\omega} - \frac{2\pi \boldsymbol{n}}{T}\right)$$

- Two-dimensional sampling formula

$$f_T(\boldsymbol{x}) = f(\boldsymbol{x})\, \text{Ш}_T(\boldsymbol{x}) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \hat{f}_T(\boldsymbol{x}) = \frac{1}{(2\pi)^2} (\hat{f} * \widehat{\text{Ш}}_T)(\boldsymbol{\omega})$$

$$\Longrightarrow \quad \hat{f}_T(\boldsymbol{x}) = \frac{1}{T^2} \sum_{\boldsymbol{n} \in \mathbb{Z}^2} \hat{f}\left(\boldsymbol{\omega} - \frac{2\pi \boldsymbol{n}}{T}\right)$$

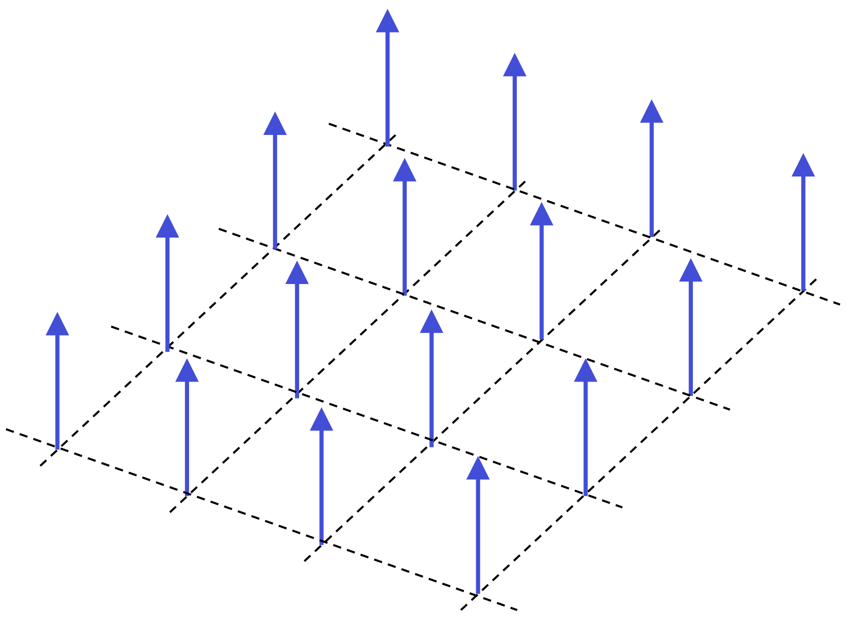
- Condition for perfect recovery (two-dimensional sampling theorem)

$$\omega_{\max} < \frac{\pi}{T} \qquad f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^2} f(\boldsymbol{k}T)\, \operatorname{sinc}\left(\frac{\boldsymbol{x} - \boldsymbol{k}T}{T}\right)$$

with $\operatorname{sinc}(\boldsymbol{x}) = \operatorname{sinc}(x_1)\operatorname{sinc}(x_2)$

Sampling Lattices

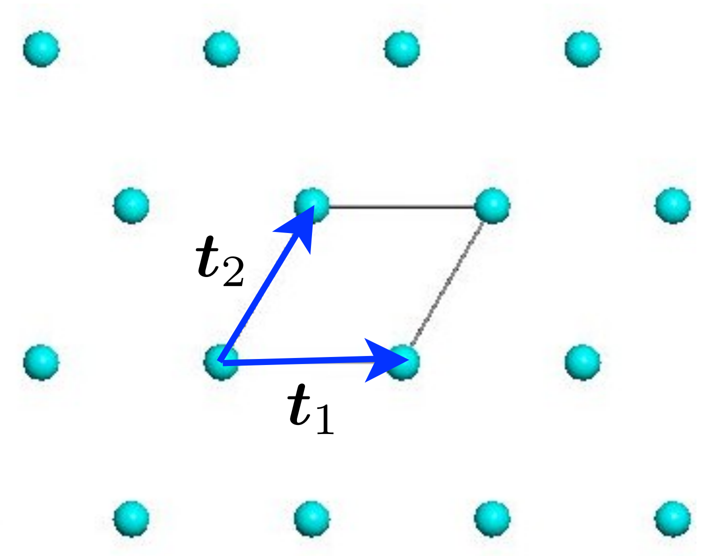
- Cartesian lattice: $\mathbb{Z}^2$



$$\boldsymbol{t}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{t}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Are there other options for sampling lattices?

- General lattice



A lattice is completely determined by two vectors: **lattice vectors**

What are the lattice vectors of the Cartesian grid?

Sampling Lattices (cont'd)

- Lattice matrix: $\mathbf{T} = \begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, with lattice vectors $\boldsymbol{t}_1$, $\boldsymbol{t}_2$

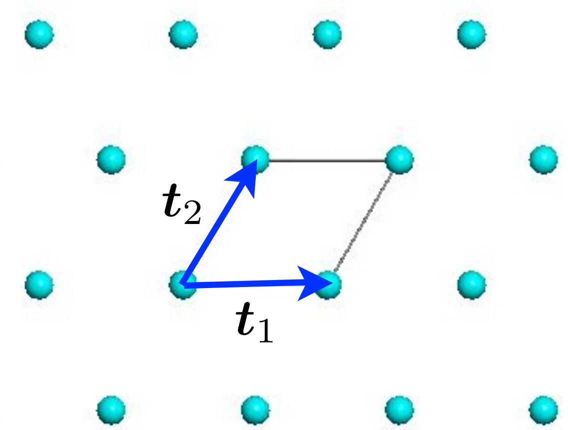
How do we use this matrix to define a lattice?

What is the lattice matrix of the Cartesian lattice?

$$\text{Cartesian lattice} \implies \mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} \text{ (identity matrix)}$$

$$\text{Cartesian lattice} \implies \mathbb{Z}^2$$

- A general lattice is of the form $\mathbf{T}\mathbb{Z}^2 = \left\{ \mathbf{T} \begin{bmatrix} m \\ n \end{bmatrix} : \begin{bmatrix} m \\ n \end{bmatrix} \in \mathbb{Z}^2 \right\}$

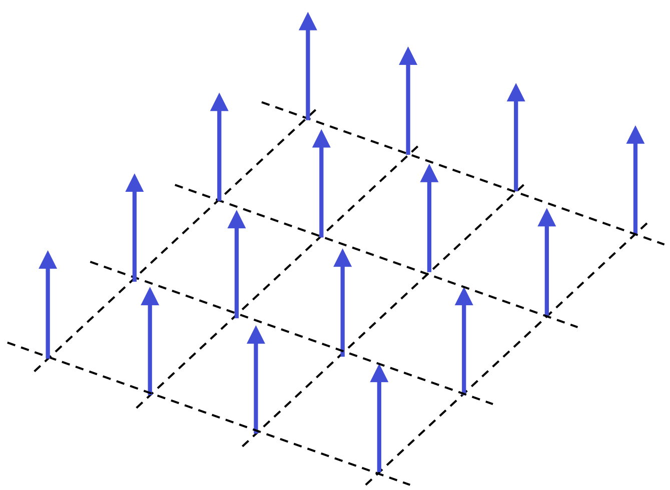


Sampling Lattices (cont'd)

What would be the Dirac comb for a general lattice?

- Dirac comb for $\mathbb{Z}^2$

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^2} \delta(\boldsymbol{x}-\boldsymbol{k}) \quad \xleftrightarrow{\mathcal{F}} \quad (2\pi)^2 \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \delta(\boldsymbol{\omega}-2\pi\boldsymbol{n})$$



Affine transformation: $f(\mathbf{T}\boldsymbol{x}) \xleftrightarrow{\mathcal{F}} \dfrac{1}{|\det(\mathbf{T})|}\hat{f}(\mathbf{T}^{-\intercal}\boldsymbol{\omega})$

- Dirac comb for $\mathbf{T}\mathbb{Z}^2$

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^2} \delta(\boldsymbol{x}-\mathbf{T}\boldsymbol{k}) \quad \xleftrightarrow{\mathcal{F}} \quad \frac{(2\pi)^2}{|\det(\mathbf{T})|} \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \delta(\boldsymbol{\omega}-2\pi\mathbf{T}^{-\intercal}\boldsymbol{n})$$

$\mathbf{T}\mathbb{Z}^2$

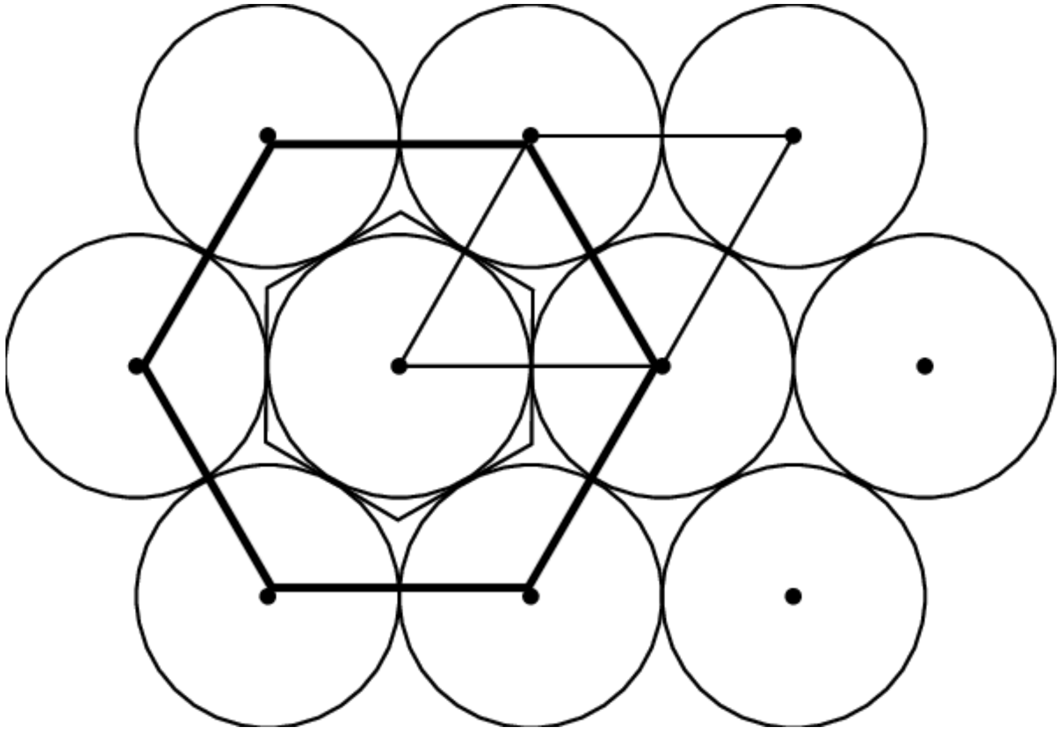
$2\pi\mathbf{T}^{-\intercal}\mathbb{Z}^2$ "dual" or "reciprocal" lattice

Sampling Lattices (cont'd)

- Adapted Poisson summation formula

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^2} f(\mathbf{T}\boldsymbol{k})\mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\mathbf{T}\boldsymbol{x}} = \frac{1}{|\det(\mathbf{T})|}\sum_{\boldsymbol{n}\in\mathbb{Z}^2} \hat{f}(\boldsymbol{\omega} - 2\pi\mathbf{T}^{-\mathsf{T}}\boldsymbol{n})$$

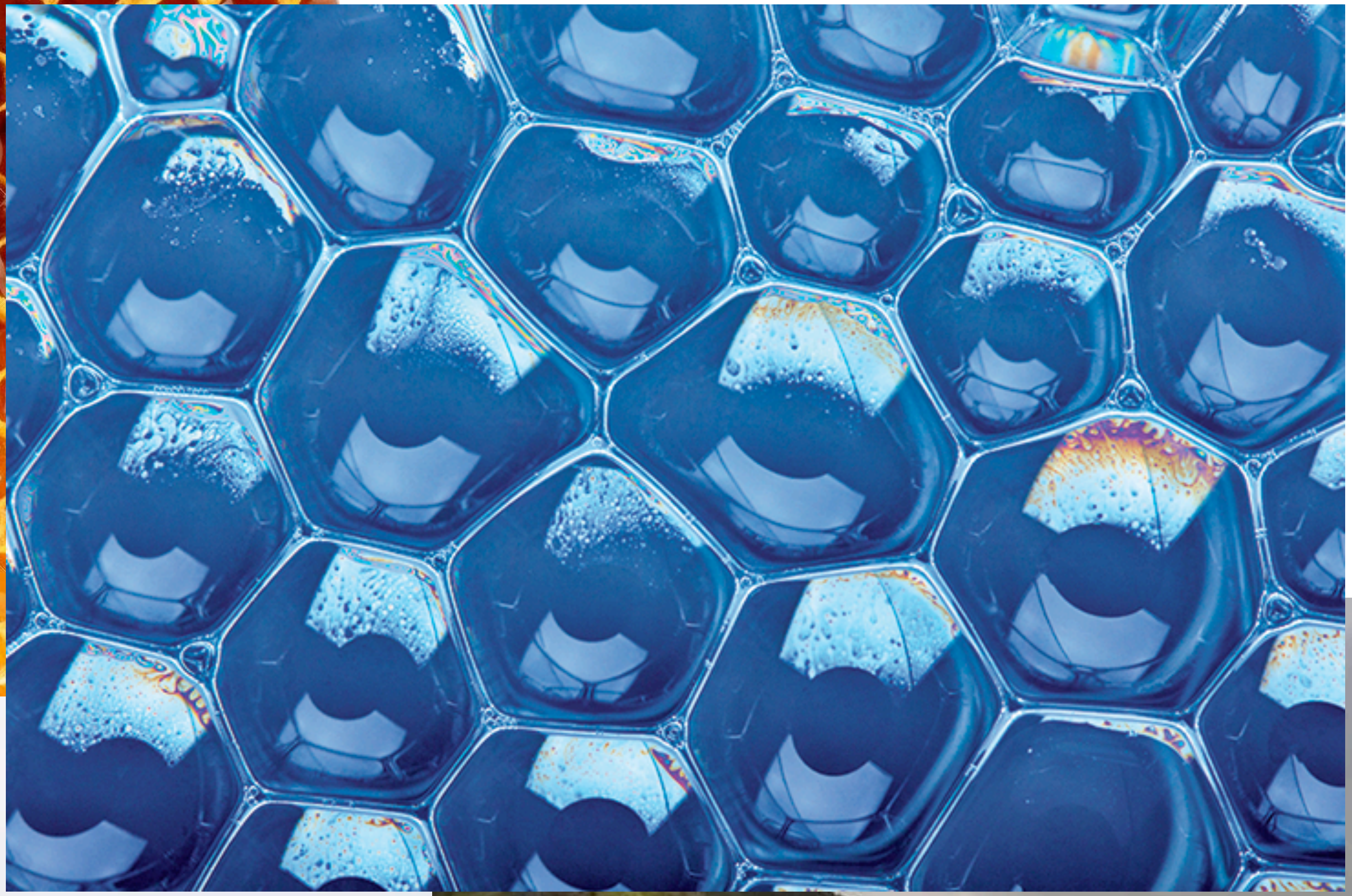
What is a "good" choice of 2D lattice?

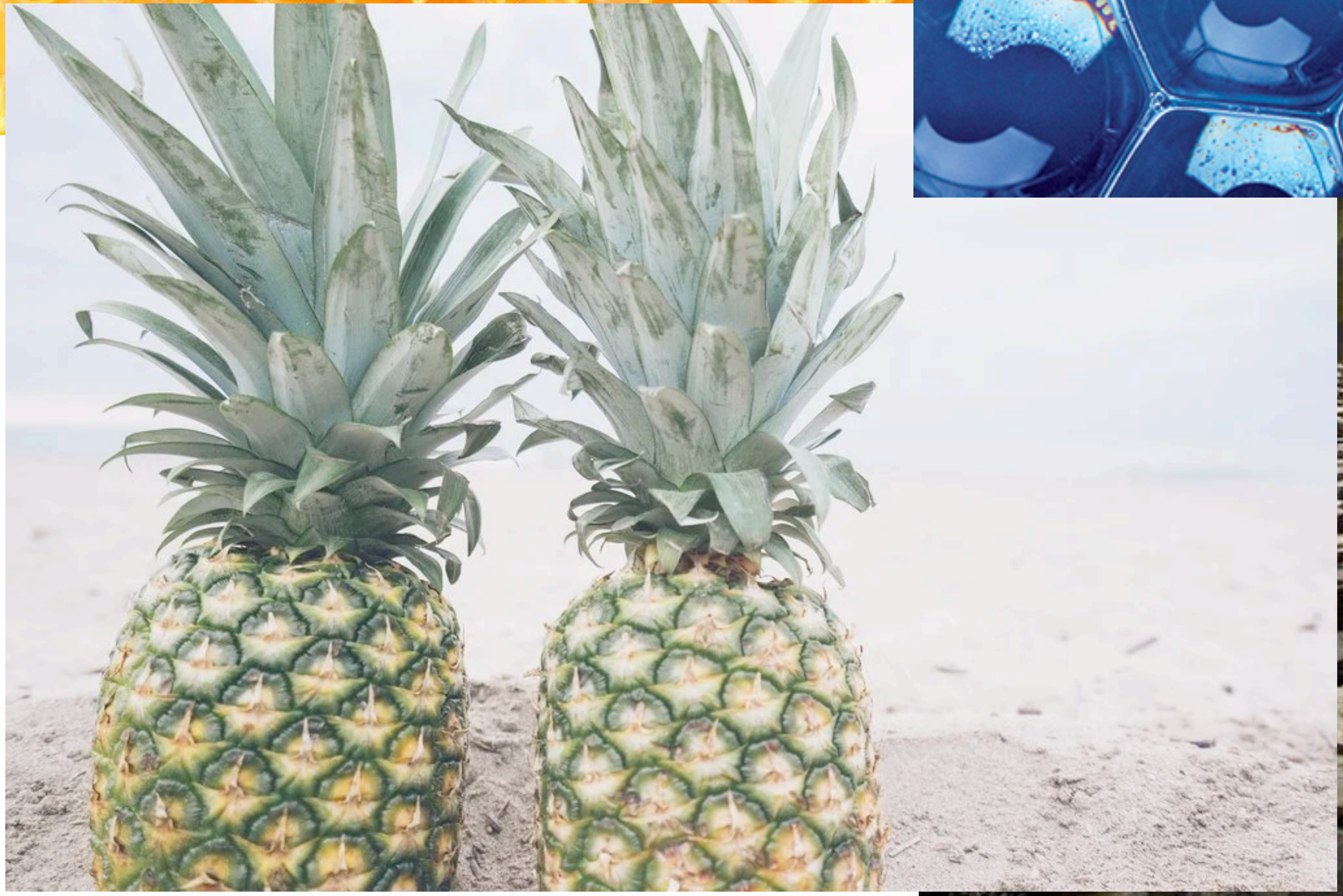


2D "sphere packing" corresponds to what lattice?

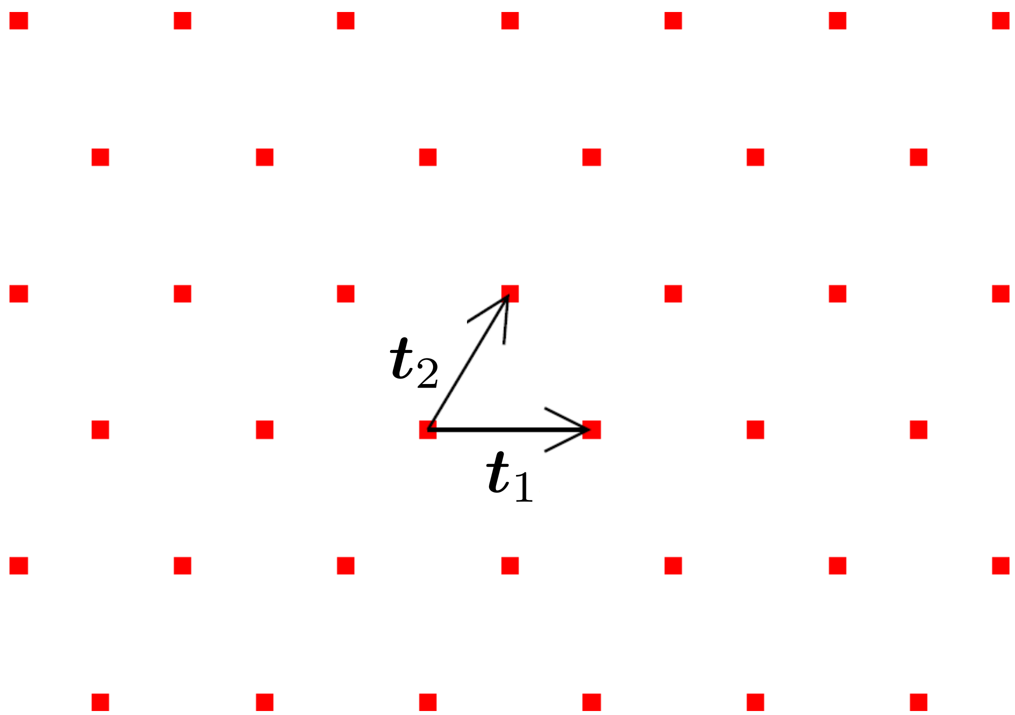
Hexagonal lattice

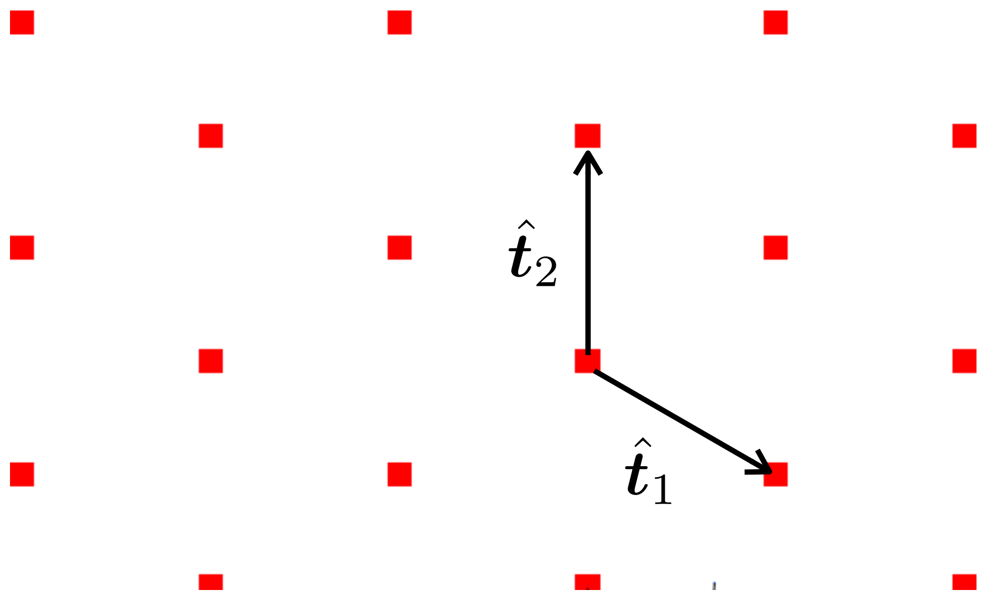
Nature's Favorite Lattice





Hexagonal Lattice





Exercise: Determine the lattice matrix.

Exercise: Determine the reciprocal lattice.

$$\mathbf{T} = \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}$$

$$2\pi \mathbf{T}^{-\mathsf{T}} = 2\pi \begin{bmatrix} 1 & 0 \\ -\sqrt{3}/3 & 2\sqrt{3}/3 \end{bmatrix}$$

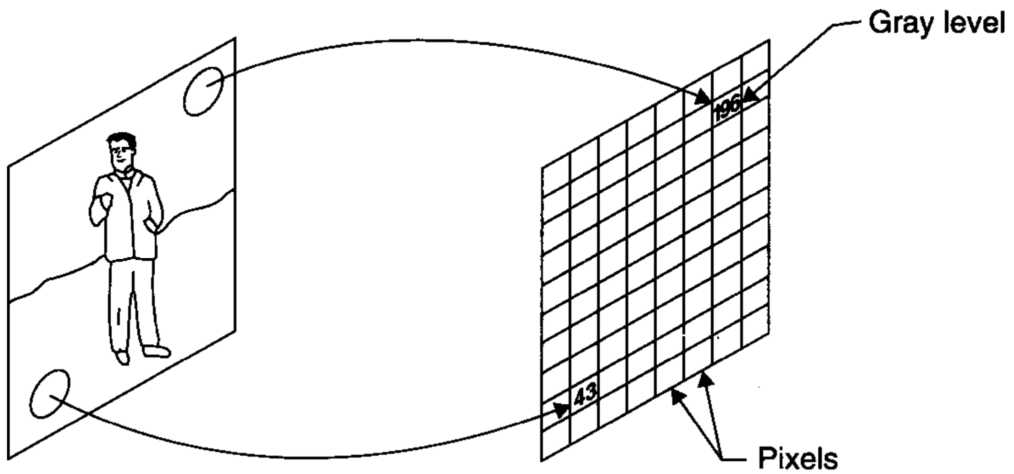
The reciprocal lattice of a hexagonal lattice is a hexagonal lattice

Acquisition Systems

- Real Acquisition Systems
- Acquisition Models
- Aliasing Problems

Real Acquisition Systems

In practice, sampling is **not ideal**



Analog image   Discrete image

- Pixel measurement process

$$p[\boldsymbol{k}] = \int_{\mathbb{R}^2} \varphi_a(\boldsymbol{y} - \boldsymbol{k}T) f(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}$$

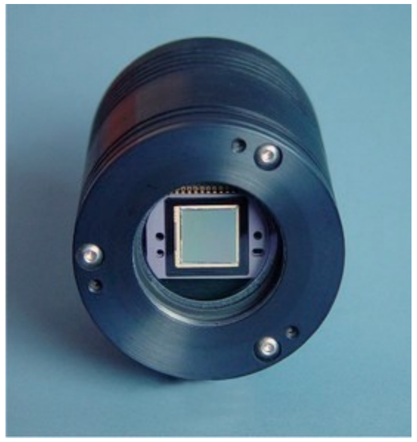
$p[\boldsymbol{k}]$: pixel value at location $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}^2$

φ_a: sampling aperture (or integration window)

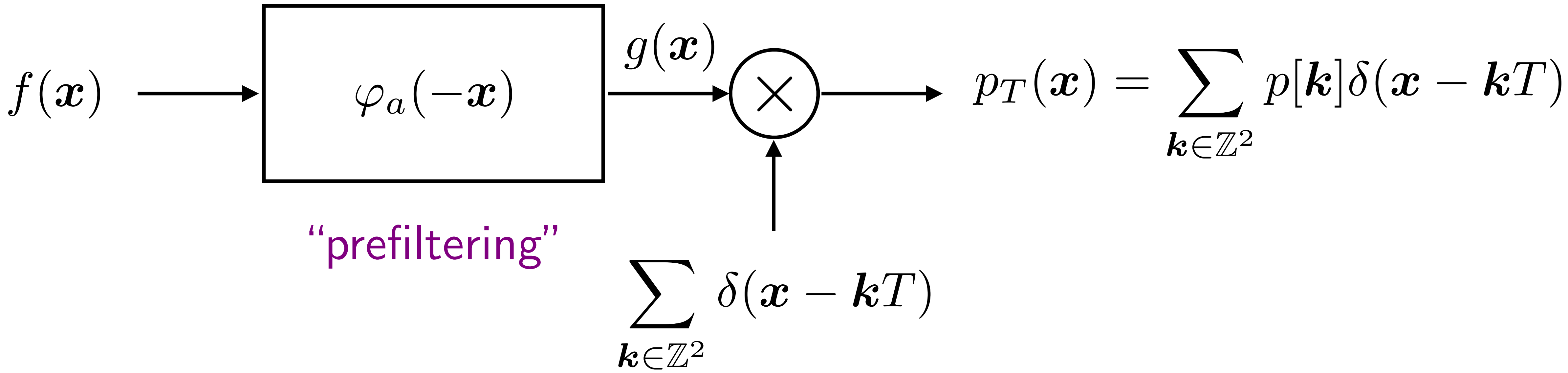
Typically: $\int_{\mathbb{R}^2} \varphi_a(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = 1$ (normalized intensity)

- Example: CCD camera

$$\varphi_a(x, y) = \frac{1}{T^2} \mathrm{rect}\left(\frac{x}{T}\right) \mathrm{rect}\left(\frac{y}{T}\right)$$



Equivalent Pixel Measurement Model



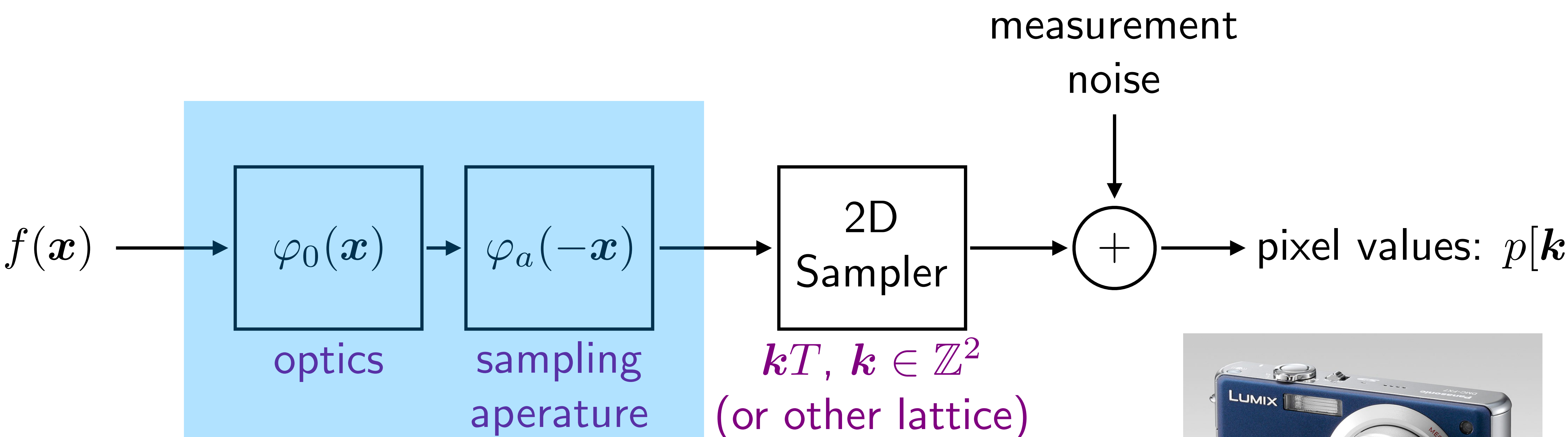
Exercise: Show that this system implements the pixel measurement process.

$$p[\boldsymbol{k}] = \int_{\mathbb{R}^2} \varphi_a(\boldsymbol{y} - \boldsymbol{k}T) f(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}$$

How does this system recover ideal sampling?

$$\varphi_a(\boldsymbol{x}) = \delta(\boldsymbol{x})$$

Even More Real Acquisition Systems

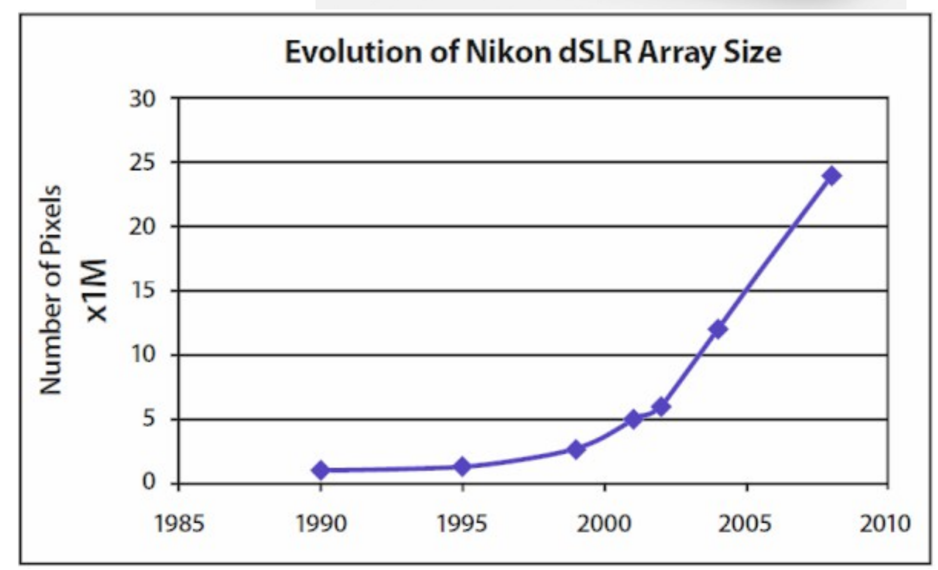


Equivalent impulse response: $h(\boldsymbol{x})$

- Complete image-acquisition model:

$$p[\boldsymbol{k}] = (h * f)(\boldsymbol{k}T) + n[\boldsymbol{k}]$$

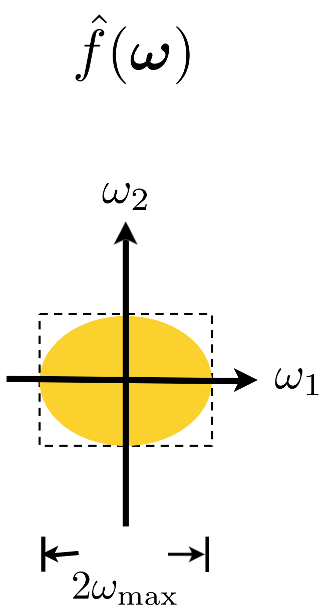
  - $\varphi_0(\boldsymbol{x})$: point-spread function (image formation)
  - $\varphi_a(\boldsymbol{x})$: sampling aperature (sensors)
  - $h(\boldsymbol{x}) = (\varphi_a^\vee * \varphi_0)(\boldsymbol{x})$: equivalent LSI system ($\varphi_a^\vee(\boldsymbol{x}) = \varphi_a(-\boldsymbol{x})$)
  - $n[\boldsymbol{k}]$: additive measurement noise

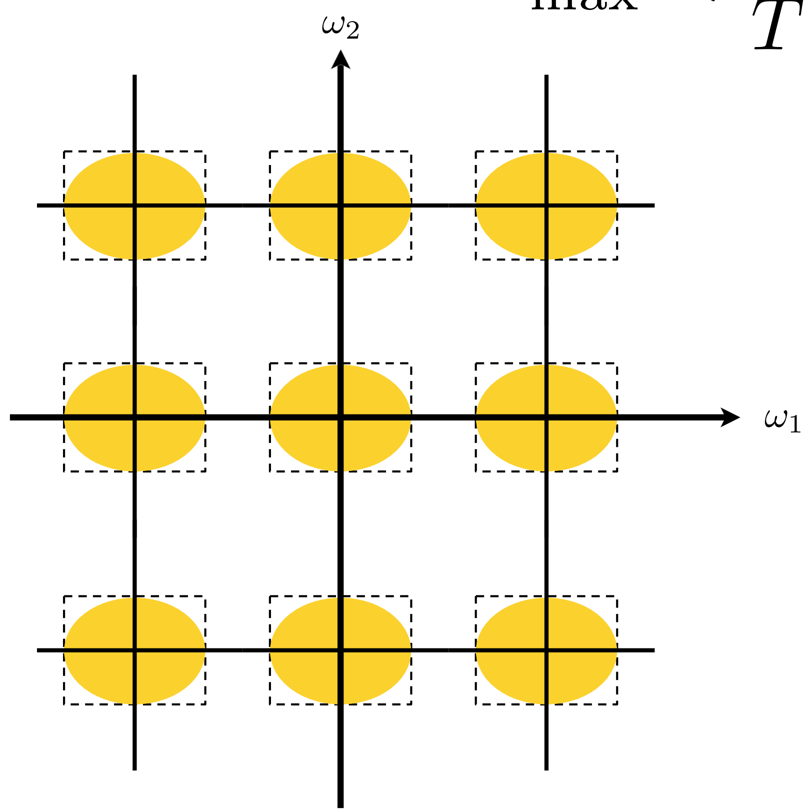


[source: dvinfo.net]

Aliasing Problems

- Aliasing = Spectral overlap induced by sampling
  - Alias-free condition: f must be bandlimited with $\omega_{\max} < \dfrac{\pi}{T}$





- Practical solutions?
  - Adapt the sampling step to the frequency content
  - Low-pass filtering prior to sampling (implicitly or explicitly)

    via φ_0: imaging system;
    φ_a: sampling aperature

Example of Aliased Image

original "analog" image
(high-resolution digital image)

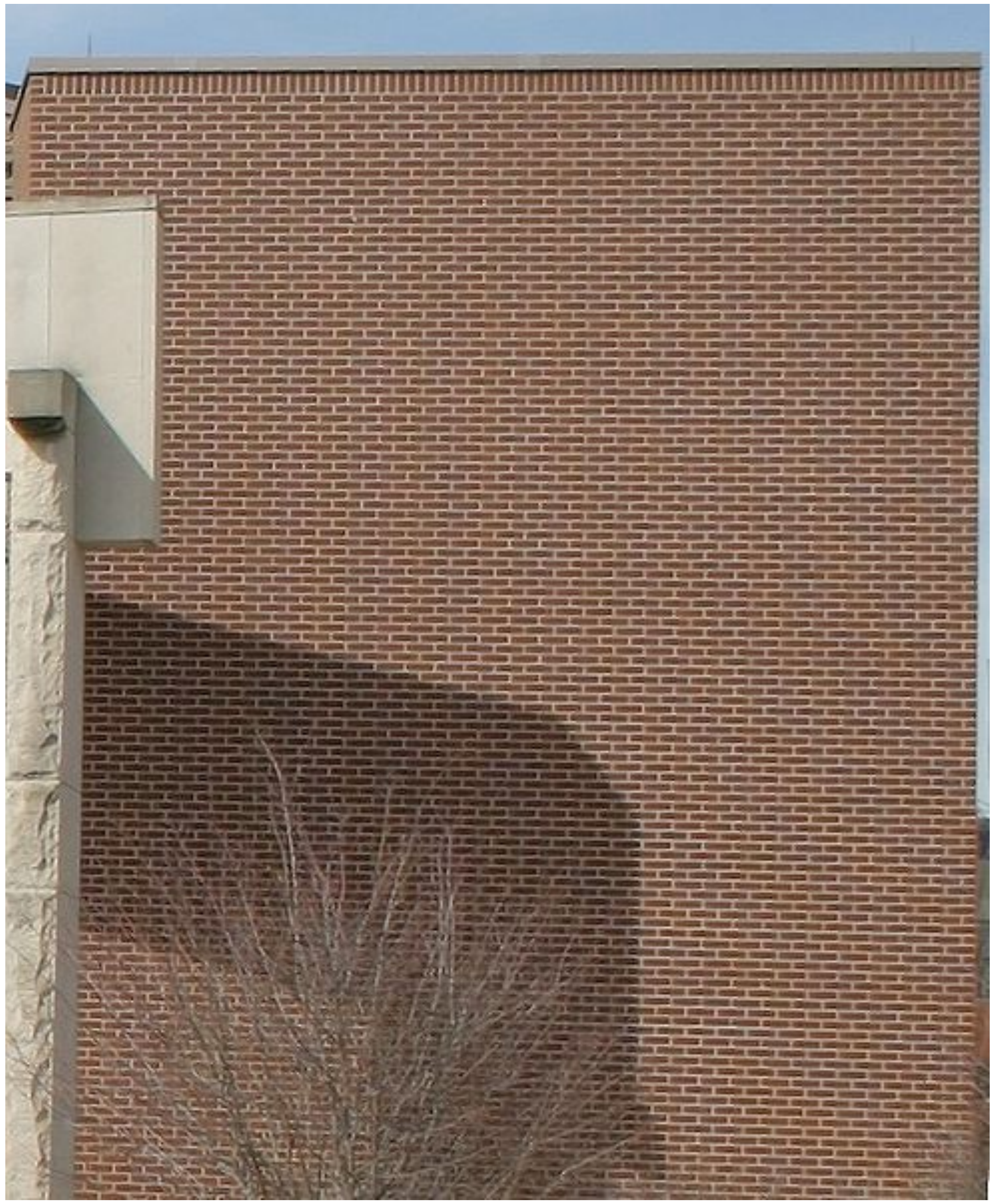
What will aliasing look like?

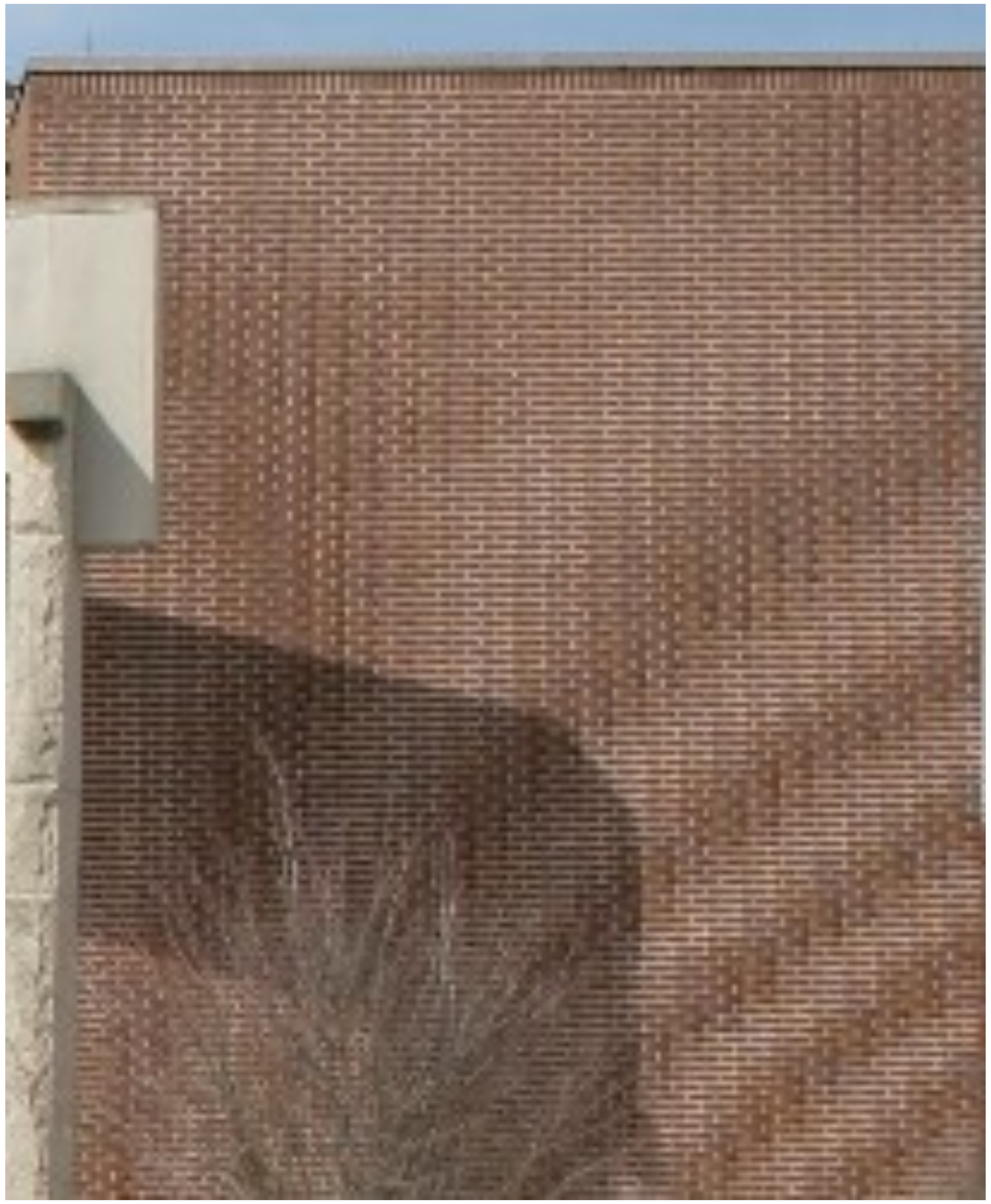
How do we simulate sampling?

sampled image
(2×2 downsampling)

prefiltered then sampled image

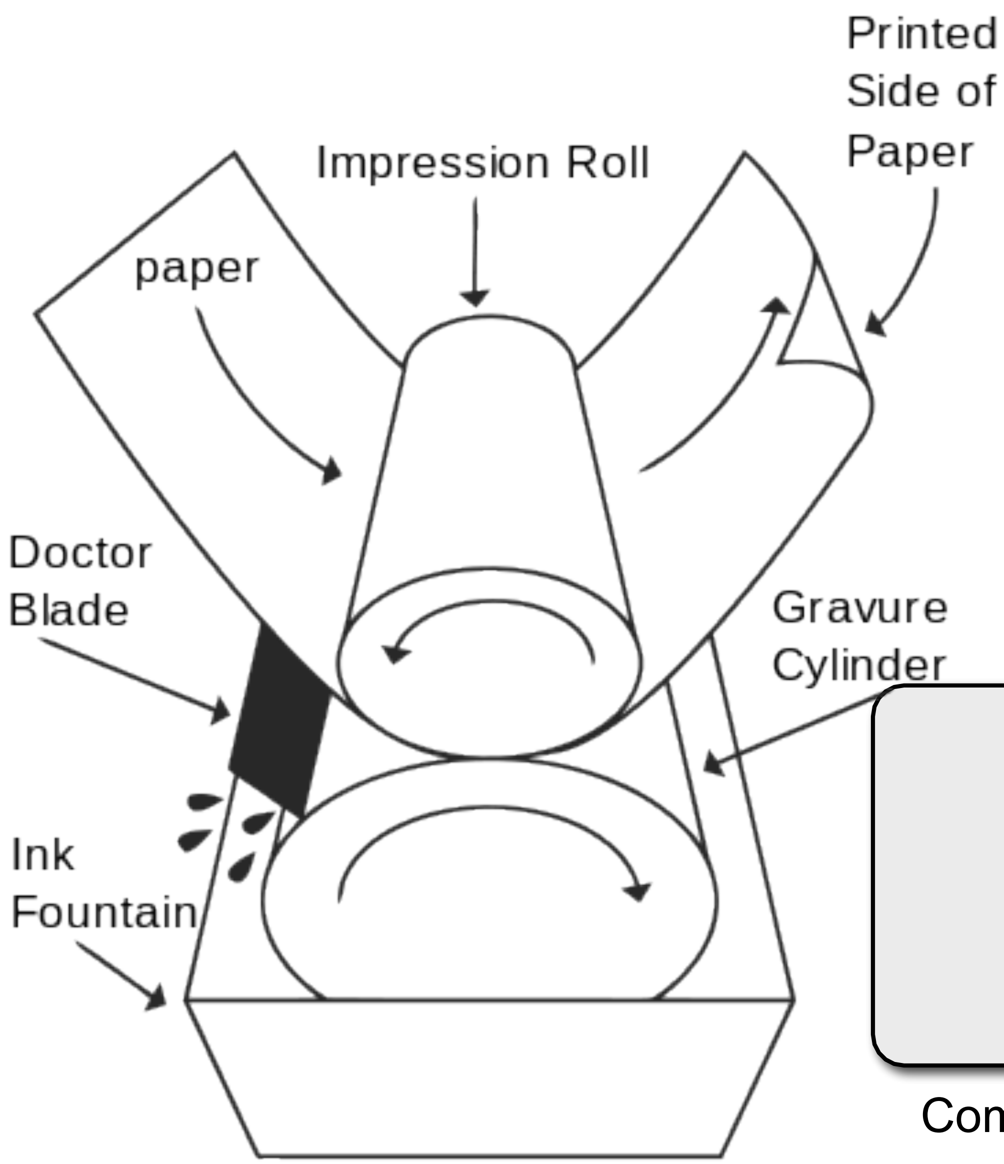
More “Bad” Aliasing

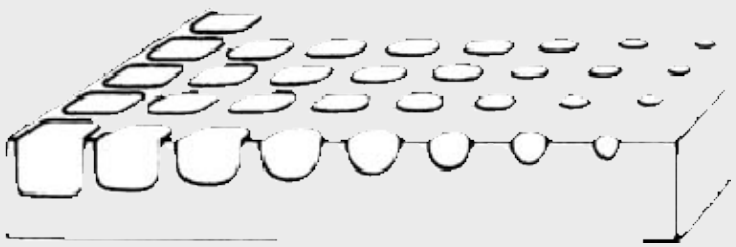




Especially a problem in printing applications

Rotogravure/Intaglio Printing





etching with diamond to make cells
with varying size and/or depth

Commonly used for mass-printed materials

This is how banknotes, passports, and stamps are printed today

Aliasing Can Be “Good”