ECE 172A: Introduction to Image Processing
Discrete Images and Filtering: Part I

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Outline

- Characterization of Discrete Images
  - Discrete Image Representation
  - Discrete-Space Fourier Transform
  - Two-Dimensional $\boldsymbol{z}$-transform ($= (z_1, z_2)$-transform)
- Discrete (Digital?) Filtering
  - Filtering With 2D Masks
  - Equivalent Filter Characterizations
  - Separability
- Filtering Images: Practical Considerations

Characterization of Discrete Images

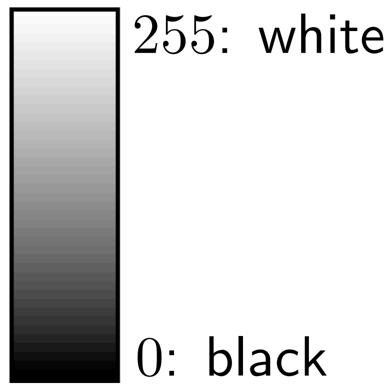
- Discrete Image Representation
- Space of Square-Summable Sequences
- Discrete-Space Fourier Transform
- Parseval/Plancherel Relation
- Two-Dimensional z-Transform
- z-Transform Properties

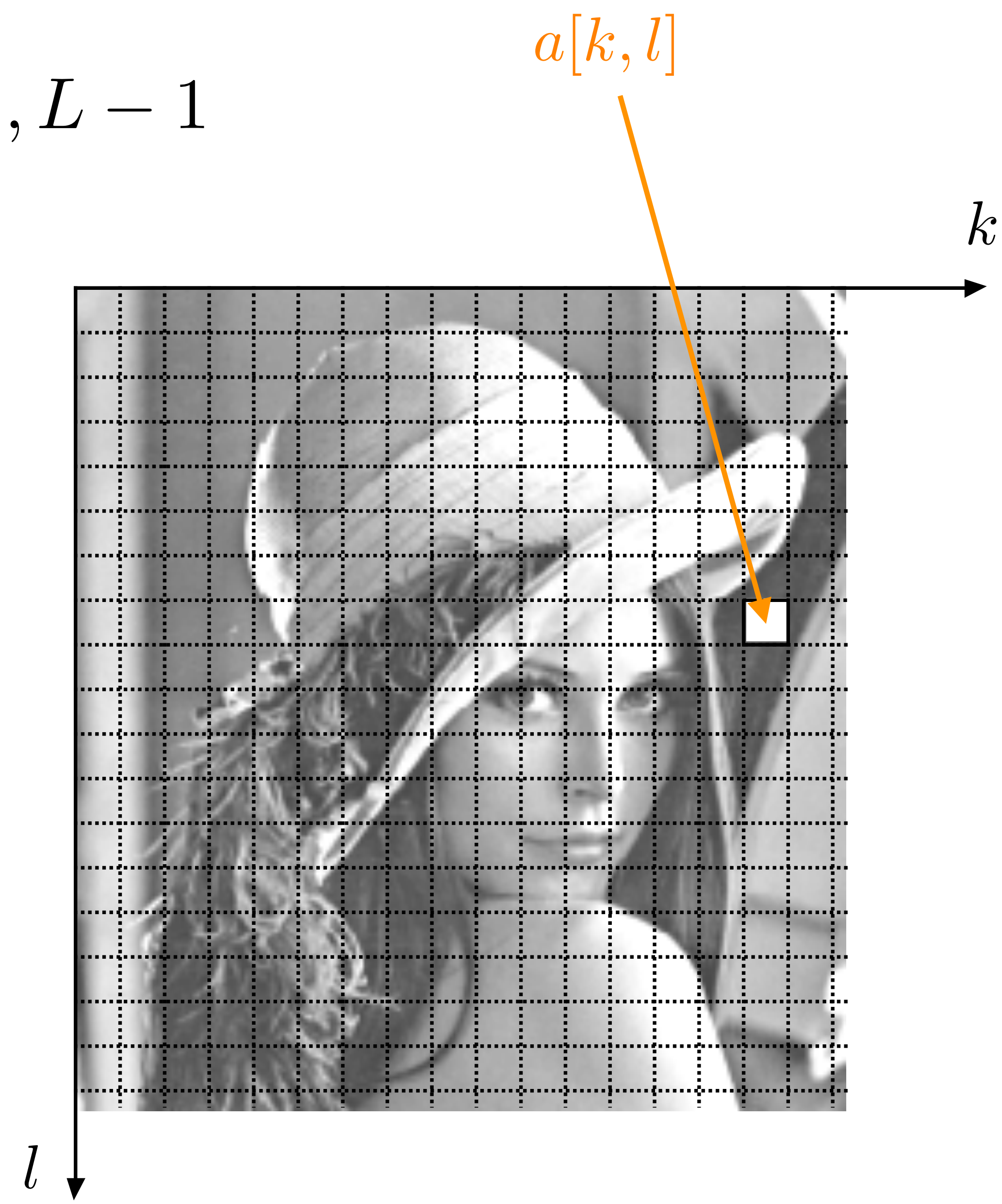
Discrete Image Representation

- Set of **pixels** $\rightarrow$ picture elements

  $\{a[k,l]\}$ with $k = 0, \ldots, K-1$ and $l = 0, \ldots, L-1$

  K: number of columns
  L: number of rows





- Array of pixels of size $K \times L$

  Storage as an $L \times K$ Python array: $\mathbf{A} = [a_{i,j}]$ with $a_{i,j} = a[j,i]$

Lab 0

Vector Space of Square-Summable Images

- View images as 2D sequences of the space variables

  $a[k_1, k_2] \in \ell^2(\mathbb{Z}^2)$ or simply $a \in \ell^2(\mathbb{Z}^2)$

  $a[\boldsymbol{k}]$ with $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}^2$ (compact vector notation)

- 2D ℓ^2-inner product

$$\langle a, b \rangle = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} a[k_1, k_2] b^*[k_1, k_2]$$

induced ℓ^2-norm:

$$\|a\|_{\ell^2} = \sqrt{\sum_{(k_1,k_2) \in \mathbb{Z}^2} |a[k_1, k_2]|^2} = \sqrt{\langle a, a \rangle}$$

- Vector space of square-summable (discrete) images

$$\ell^2(\mathbb{Z}^2) = \{a[\boldsymbol{k}] \ : \ \boldsymbol{k} \in \mathbb{Z}^2 \text{ and } \|a\|^2_{\ell^2} < \infty\}$$

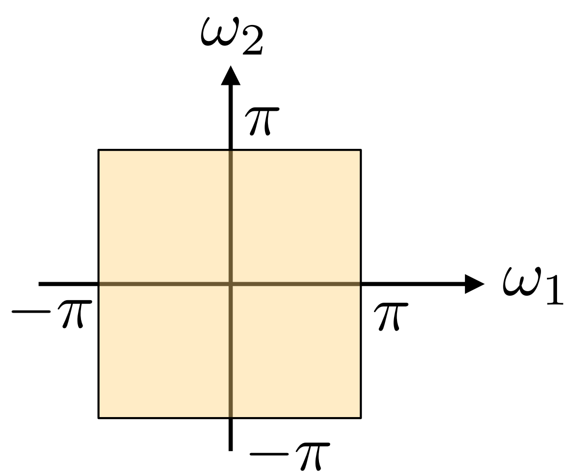
Discrete-Space Fourier Transform

- 2D discrete-space Fourier transform: Definition

$$\hat{a}(\omega_1, \omega_2) = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} a[k_1, k_2] \mathrm{e}^{-\mathrm{j}(\omega_1 k_1 + \omega_2 k_2)} \quad \text{with} \quad (\omega_1, \omega_2) \in \mathbb{R}^2$$

- $(2\pi \times 2\pi)$-periodicity

$$\hat{a}(\omega_1, \omega_2) = \hat{a}(\omega_1 + m2\pi, \omega_2 + n2\pi), \quad (m, n) \in \mathbb{Z}^2$$



Support of the **main** Fourier period:
$[-\pi, \pi]^2 = [-\pi, \pi] \times [-\pi, \pi]$

- Inverse transform

$$a[k_1, k_2] = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \hat{a}(\omega_1, \omega_2) \mathrm{e}^{\mathrm{j}(\omega_1 k_1 + \omega_2 k_2)} \, \mathrm{d}\omega_1 \, \mathrm{d}\omega_2$$

Parseval-Plancherel Relations

- Discrete-Space Fourier transform and finite energy:

$$a \in \ell^2(\mathbb{Z}^2) \quad \text{if and only if} \quad \hat{a} \in L^2([-\pi,\pi]^2)$$

Theorem: The complex exponentials $\{\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^2}$ form an **orthonormal basis** of $L^2([-\pi,\pi]^2)$ with respect to the inner product $\langle \hat{a}, \hat{b}\rangle = \dfrac{1}{(2\pi)^2}\displaystyle\int_{[-\pi,\pi]^2} \hat{a}(\boldsymbol{\omega})\hat{b}^*(\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega}$.

Dual interpretation via Fourier series:

- $\hat{a}(\boldsymbol{\omega}) = \displaystyle\sum_{\boldsymbol{k}\in\mathbb{Z}^2} a[\boldsymbol{k}]\mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{k}}$ is the **Fourier series** of $\hat{a}(\boldsymbol{\omega})$
- $a[\boldsymbol{k}] = \langle \hat{a}(\boldsymbol{\omega}), \mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{k}}\rangle$ are the **Fourier coefficients**

Parseval-Plancherel Relations

- Discrete-Space Fourier transform and finite energy:

$$a \in \ell^2(\mathbb{Z}^2) \quad \text{if and only if} \quad \hat{a} \in L^2([-\pi,\pi]^2)$$

- Parseval's formula: $\langle a, b\rangle = \langle \hat{a}, \hat{b}\rangle$

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^2} a[\boldsymbol{k}]b^*[\boldsymbol{k}] = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} \hat{a}(\boldsymbol{\omega})\hat{b}^*(\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega}$$

- Energy-preservation property:

$$\|a\|_{\ell^2}^2 = \|\hat{a}\|_{L^2([-\pi,\pi]^2)}^2 = \frac{1}{(2\pi)^2}\int_{[-\pi,\pi]^2} |\hat{a}(\boldsymbol{\omega})|^2\,\mathrm{d}\boldsymbol{\omega}$$

Relationship With Continuous Fourier Transform

- Representation of bandlimited functions:

$$f_{\text{int}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} f[\boldsymbol{k}]\,\mathrm{sinc}(\boldsymbol{x}-\boldsymbol{k})$$

Exercise: What is the maximum frequency content of f for this to hold?

$$\begin{aligned}
\hat{f}_{\text{int}}(\boldsymbol{\omega}) &= \sum_{\boldsymbol{k}\in\mathbb{Z}^2} f[\boldsymbol{k}]\,\mathcal{F}\{\mathrm{sinc}(\boldsymbol{x}-\boldsymbol{k})\}(\boldsymbol{\omega}) \\
&= \boxed{\sum_{\boldsymbol{k}\in\mathbb{Z}^2} f[\boldsymbol{k}]\,\mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\top\boldsymbol{k}}}\,\mathcal{F}\{\mathrm{sinc}(\boldsymbol{x})\}(\boldsymbol{\omega}) \\
&= \hat{f}_{\mathrm{d}}(\boldsymbol{\omega})\,\mathrm{rect}\left(\frac{\boldsymbol{\omega}}{2\pi}\right) \qquad = \begin{cases} \hat{f}_{\mathrm{d}}(\boldsymbol{\omega}), & \text{for } \boldsymbol{\omega}\in[-\pi,\pi]^2 \\ 0, & \text{else} \end{cases}
\end{aligned}$$

The representation holds when f_{int} is bandlimited to $[-\pi,\pi]^2$

Two-Dimensional z-Transform

Complex variable: $\boldsymbol{z} = (z_1, z_2) \in \mathbb{C}^2$ Space index: $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}^2$

- Definitions:
  - Multi-index notation: $\boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} z_2^{k_2}$
  - $\boldsymbol{z}$-transform (or (z_1, z_2)-transform): $A(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^2} a[\boldsymbol{k}] \boldsymbol{z}^{-\boldsymbol{k}}$

    (for $\boldsymbol{z}$ in the ROC)

- Relationship with discrete-space Fourier transform:

  $z_1 = \mathrm{e}^{\mathrm{j}\omega_1}$ and $z_2 = \mathrm{e}^{\mathrm{j}\omega_2}$ (restriction to unit circle)

  Define: $\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}} = (\mathrm{e}^{\mathrm{j}\omega_1}, \mathrm{e}^{\mathrm{j}\omega_2})$

  $$A(\boldsymbol{z})\big|_{\boldsymbol{z}=\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}}} = \hat{a}(\boldsymbol{\omega})$$

Region of Convergence

Definition: The ROC is the subset of $\mathbb{C}^2$ for which
$$A(\boldsymbol{z}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} a[\boldsymbol{k}]\boldsymbol{z}^{-\boldsymbol{k}} \text{ converges.}$$

- Practical constraint: Discrete-space Fourier transform converges

  $\Longrightarrow$ ROC must include the unit circle: $z_1 = \mathrm{e}^{\mathrm{j}\omega_1}$ and $z_2 = \mathrm{e}^{\mathrm{j}\omega_2}$

- Most cases fall into two categories where this holds:
  - $a[\boldsymbol{k}]$ is bounded with finite support (FIR) ROC $= \mathbb{C}^2 \setminus \{\mathbf{0}\}$
  - $a[\boldsymbol{k}]$ is **absolutely summable** ROC contains the unit circle

z-Transform Properties

- Separability: $$f[\boldsymbol{k}] = f_1[k_1]f_2[k_2] \overset{\boldsymbol{z}}{\longleftrightarrow} F(\boldsymbol{z}) = F_1(z_1)F_2(z_2)$$
- Delay/Shift: $$f[\boldsymbol{k} - \boldsymbol{k}_0] \overset{\boldsymbol{z}}{\longleftrightarrow} \boldsymbol{z}^{-\boldsymbol{k}_0} F(\boldsymbol{z})$$
- Reflection: $$f^{\vee}[\boldsymbol{k}] = f[-\boldsymbol{k}] \overset{\boldsymbol{z}}{\longleftrightarrow} F(z_1^{-1}, z_2^{-1})$$
- Convolution: $$g[\boldsymbol{k}] = (h * f)[\boldsymbol{k}] = \sum_{\boldsymbol{n} \in \mathbb{Z}^2} h[\boldsymbol{n}]f[\boldsymbol{k} - \boldsymbol{n}] \overset{\boldsymbol{z}}{\longleftrightarrow} G(\boldsymbol{z}) = H(\boldsymbol{z})F(\boldsymbol{z})$$

Proof of the Convolution Theorem: $G(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^2} \sum_{\boldsymbol{n} \in \mathbb{Z}^2} h[\boldsymbol{n}]f[\boldsymbol{k} - \boldsymbol{n}]\boldsymbol{z}^{-\boldsymbol{k}}$

change of variables: $\boldsymbol{m} = \boldsymbol{k} - \boldsymbol{n}$ $\quad = \sum_{\boldsymbol{m} \in \mathbb{Z}^2} \sum_{\boldsymbol{n} \in \mathbb{Z}^2} h[\boldsymbol{n}]f[\boldsymbol{m}]\boldsymbol{z}^{-(\boldsymbol{n}+\boldsymbol{m})}$

$$= \left(\sum_{\boldsymbol{n} \in \mathbb{Z}^2} h[\boldsymbol{n}]\boldsymbol{z}^{-\boldsymbol{n}} \right) \left(\sum_{\boldsymbol{m} \in \mathbb{Z}^2} f[\boldsymbol{m}]\boldsymbol{z}^{-\boldsymbol{m}} \right)$$

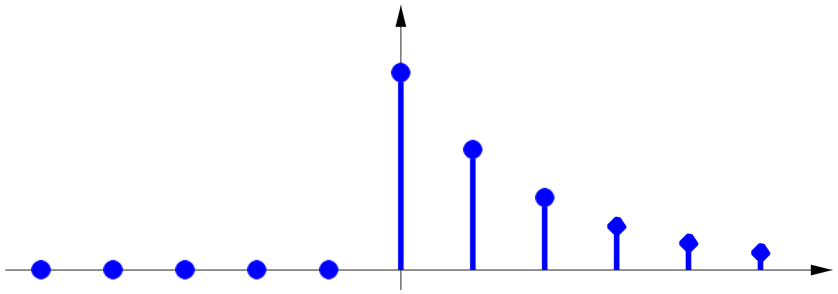
$$= H(\boldsymbol{z})F(\boldsymbol{z})$$

1D z-Transform Examples

- Definition: $H(z) = \sum_{k \in \mathbb{Z}} h[k] z^{-k}$

- Causal exponential

$$h_+[k] = \begin{cases} a^k, & k \geq 0 \\ 0, & \text{else} \end{cases} \quad \text{with} \quad 0 < |a| < 1$$

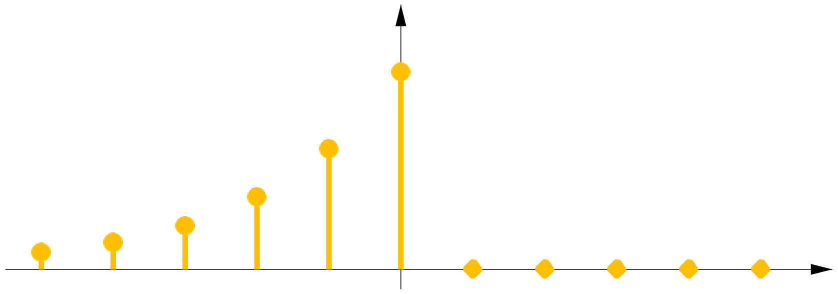


Exercise: Determine $H_+(z) = \dfrac{1}{1 - az^{-1}}$

(pole inside the unit circle)

- Anti-causal exponential

$$h_-[k] = \begin{cases} a^{|k|}, & k \leq 0 \\ 0, & \text{else} \end{cases} \quad \text{with} \quad 0 < |a| < 1$$



$$H_-(z) = H_+(z^{-1}) = \frac{1}{1 - az}$$

(pole outside the unit circle)

2D z-Transform Examples

- Definition: $H(\boldsymbol{z}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} h[\boldsymbol{k}]\boldsymbol{z}^{-\boldsymbol{k}} = \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} h[k_1,k_2] z_1^{-k_1} z_2^{-k_2}$

$$h[k_1,k_2] = \begin{array}{c|ccc} & & \xrightarrow{\quad k_1 \quad} & \\ \hline \downarrow & 1 & 0 & -1 \\ k_2 & 2 & \boxed{0} & -2 \\ & 1 & 0 & -1 \end{array}$$

Exercise: Compute $H(z_1, z_2)$

$$\begin{array}{cccccccc|c} -1 & & 0 & & +1 & & k_1 \rightarrow & & k_2 \downarrow \\ (1)\cdot z_1 z_2 & + & (0)\cdot 1\cdot z_2 & + & (-1)\cdot z_1^{-1} z_2 & + & & & -1 \\ (2)\cdot z_1\cdot 1 & + & (0)\cdot 1\cdot 1 & + & (-2)\cdot z_1^{-1}\cdot 1 & + & & & 0 \\ (1)\cdot z_1 z_2^{-1} & + & (0)\cdot 1\cdot z_2^{-1} & + & (-1)\cdot z_1^{-1} z_2^{-1} & & & & 1 \end{array}$$

$$= (z_1 - z_1^{-1})(z_2 + 2 + z_2^{-1})$$

Separable filter!

Inverse z-Transform

- Identify coefficients of the polynomial: $H(\boldsymbol{z}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} h[\boldsymbol{k}]\boldsymbol{z}^{-\boldsymbol{k}}$
- Take advantage of separability: $H(\boldsymbol{z}) = H_1(z_1)H_2(z_2)$
- Reminder of 1D methods:
  - Cauchy integral theorem: $h[k] = \frac{1}{\mathrm{j}2\pi}\oint_\Gamma H(z)z^{k-1}\,\mathrm{d}z$

    Γ is any contour that encloses the origin
- More often than not, we will just use tables and/or partial fractions

E.g., $\frac{-3}{2z^{-1}-5+2z} = \frac{3/4}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z\right)} = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{2}z} - 1$

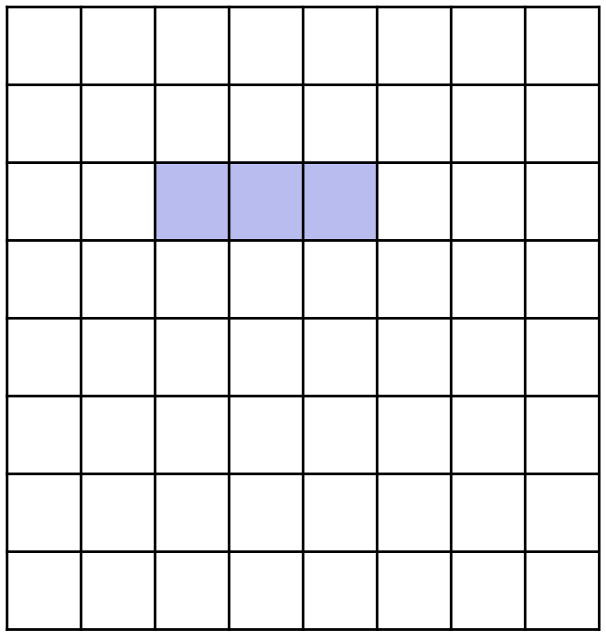
$$\Longrightarrow \quad h[k] = u[k]\left(\frac{1}{2}\right)^k + u[-k]\left(\frac{1}{2}\right)^{-k} - \delta[k]$$

Discrete/Digital Filtering

- Filtering With 2D Masks
- Linearity and Shift-Invariance
- Impulse Response and Discrete Convolution
- Equivalent Filter Characterizations
- Examples of Transfer Functions
- Separability
- z-transform and Recursive Filtering

Filtering With 2D Masks

- Mask (or local operator) formulation



  - Filtering mask (weights): $(2M+1)\times(2N+1)$

$$\boldsymbol{w} = \begin{bmatrix} w[-M,-N] & \cdots & w[M,-N] \\ \vdots & w[0,0] & \vdots \\ w[-M,N] & \cdots & w[M,N] \end{bmatrix}$$

  - Local neighborhood: $(2M+1)\times(2N+1)$

$$\boldsymbol{f_k} = \begin{bmatrix} f[k_1-M,k_2-N] & \cdots & f[k_1+M,k_2-N] \\ \vdots & f[k_1,k_2] & \vdots \\ f[k_1-M,k_2+N] & \cdots & f[k_1+M,k_2+N] \end{bmatrix}$$

- Filtering: matrix formulation

$$g[\boldsymbol{k}] = \boldsymbol{f_k} \odot \boldsymbol{w} = \sum_i \sum_j [\boldsymbol{f_k}]_{i,j} [\boldsymbol{w}]_{i,j} \qquad \text{(term-by-term product)}$$

This is called a **correlation** (not a **convolution**)

Filtering Examples

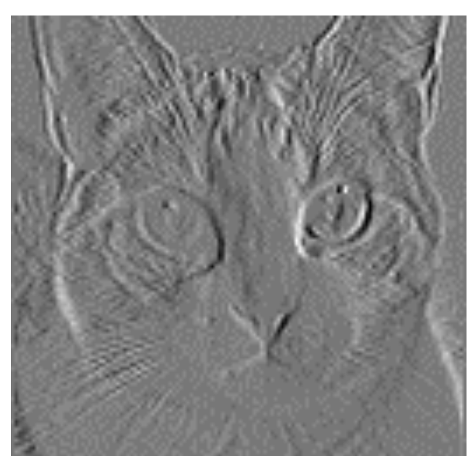
Mask: $\boldsymbol{w}$

– Local (3×3)-average

$$\boldsymbol{w}_{\text{ave}} = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \boxed{1} & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

– Horizontal-edge enhancement

$$\boldsymbol{w}_{\text{hor}} = \begin{bmatrix} -1 & -2 & -1 \\ 0 & \boxed{0} & 0 \\ 1 & 2 & 1 \end{bmatrix}$$



– Vertical-edge enhancement

$$\boldsymbol{w}_{\text{vert}} = \begin{bmatrix} -1 & 0 & 1 \\ -2 & \boxed{0} & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Linearity and Shift Invariance

- Most filters are linear:

$$\alpha f_1[\boldsymbol{k}] + \beta f_2[\boldsymbol{k}] \longrightarrow \boxed{H(\boldsymbol{z})} \longrightarrow \alpha(h * f_1)[\boldsymbol{k}] + \beta(h * f_2)[\boldsymbol{k}] \qquad \forall \alpha, \beta \in \mathbb{R}$$

- Shift by $\boldsymbol{k}_0$ operator:

$$f[\boldsymbol{k}] \longrightarrow \boxed{\boldsymbol{z}^{-\boldsymbol{k}_0}} \longrightarrow f[\boldsymbol{k} - \boldsymbol{k}_0]$$

- Most filters are shift-invariant:

$$\longrightarrow \boxed{H(\boldsymbol{z})} \longrightarrow \boxed{\boldsymbol{z}^{-\boldsymbol{k}_0}} \longrightarrow \quad \equiv \quad \longrightarrow \boxed{\boldsymbol{z}^{-\boldsymbol{k}_0}} \longrightarrow \boxed{H(\boldsymbol{z})} \longrightarrow$$

Characterization of Filters

Can we have linear filters that are not shift-invariant?

- Linear but not shift-invariant filters:

$f[\boldsymbol{k}] \longrightarrow \boxed{\text{linear filter}} \longrightarrow g[\boldsymbol{k}] = \boldsymbol{f}_{\boldsymbol{k}} \odot \boldsymbol{w}_{\boldsymbol{k}}$

spatially-varying mask

- Linear and shift-invariant filters:

$f[\boldsymbol{k}] \longrightarrow \boxed{\text{LSI filter}} \longrightarrow g[\boldsymbol{k}] = \boldsymbol{f}_{\boldsymbol{k}} \odot \boldsymbol{w} = (h * f)[\boldsymbol{k}]$

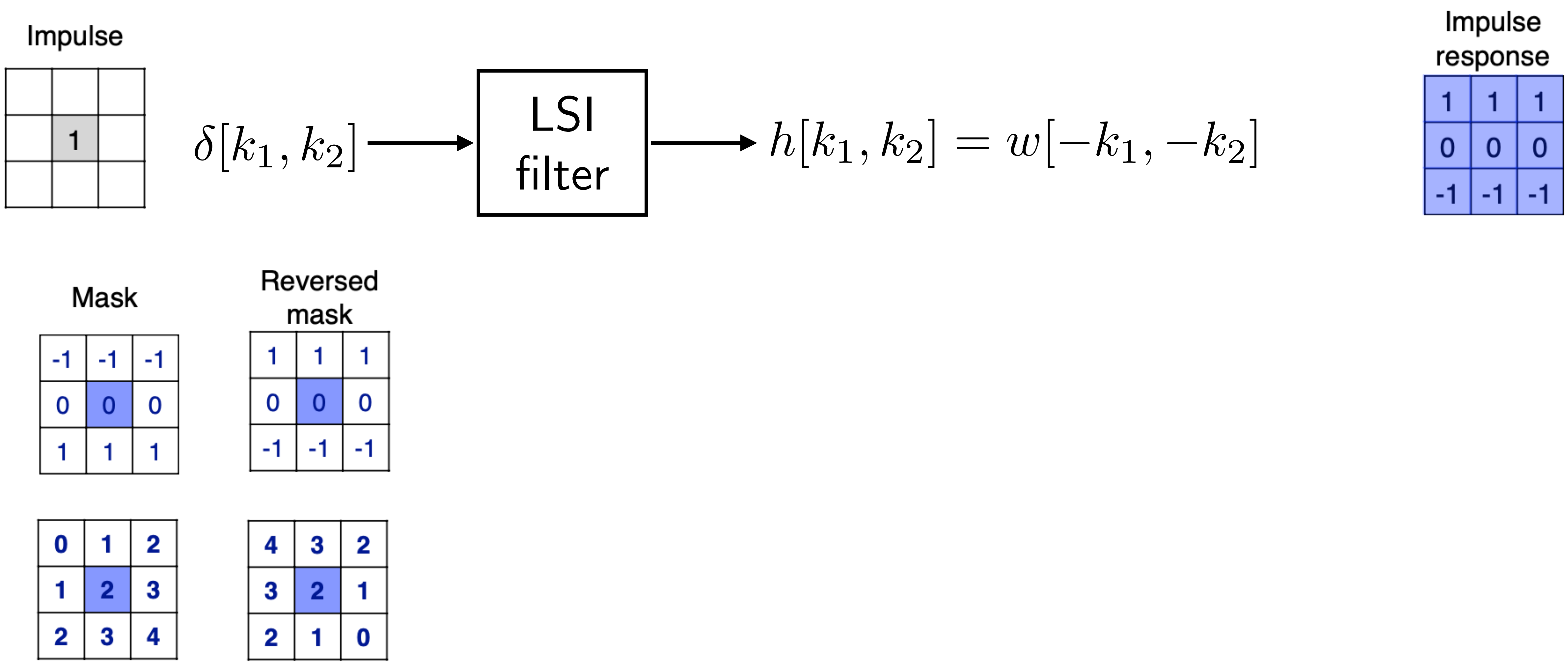
fixed mask

How do we determine the mask from the impulse response?

Masks and Impulse Responses

- Filter implementation using a mask: Correlation

$$g[k_1, k_2] = \sum_i \sum_j w[i, j] f[k_1 + i, k_2 + j]$$



Mask = space-reversed impulse response (and vice-versa)

Impulse Responses and Discrete Convolutions

- Impulse response (point-spread function)

  unit impulse (or Kronecker impulse or Kronecker delta)

  $$\delta[\boldsymbol{k}] \longrightarrow \boxed{\begin{array}{c}\text{LSI}\\ \text{filter}\end{array}} \longrightarrow h[\boldsymbol{k}] \quad \text{impulse reponse}$$

  > A discrete LSI filter is uniquely characterized by its impulse response, which is the **space-reversed** version of its mask: $h[\boldsymbol{k}] = w[-\boldsymbol{k}]$

- Implementation by convolution

  $$f[k_1, k_2] \longrightarrow \boxed{\begin{array}{c}\text{LSI}\\ \text{filter}\end{array}} \longrightarrow g[k_1, k_2] = (h * f)[k_1, k_2]$$

  $$= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} f[m, n] h[k_1 - m, k_2 - n]$$

Pixel-Wise Interpretation

- Unit impulse (pixel) at the origin

$$\delta[\boldsymbol{k}] \longrightarrow \boxed{\begin{array}{c}\text{LSI}\\ \text{filter}\end{array}} \longrightarrow h[\boldsymbol{k}]$$

- Pixel at location $\boldsymbol{k}_0$

$$\delta[\boldsymbol{k}-\boldsymbol{k}_0] \longrightarrow \boxed{\begin{array}{c}\text{LSI}\\ \text{filter}\end{array}} \longrightarrow h[\boldsymbol{k}-\boldsymbol{k}_0]$$

- Pixel at location $\boldsymbol{k}_0$ with value $f[\boldsymbol{k}_0]$

$$f[\boldsymbol{k}_0]\delta[\boldsymbol{k}-\boldsymbol{k}_0] \longrightarrow \boxed{\begin{array}{c}\text{LSI}\\ \text{filter}\end{array}} \longrightarrow f[\boldsymbol{k}_0]h[\boldsymbol{k}-\boldsymbol{k}_0]$$

- Input image $=$ sum of pixels

$$\sum_{\boldsymbol{k}_0} f[\boldsymbol{k}_0]\delta[\boldsymbol{k}-\boldsymbol{k}_0] \longrightarrow \boxed{\begin{array}{c}\text{LSI}\\ \text{filter}\end{array}} \longrightarrow \sum_{\boldsymbol{k}_0} f[\boldsymbol{k}_0]h[\boldsymbol{k}-\boldsymbol{k}_0] = (h * f)[\boldsymbol{k}]$$

Equivalent LSI Filter Characterizations

LSI filter $=$ discrete convolution operator

$$g[\boldsymbol{k}] = (h * f)[\boldsymbol{k}] = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} h[\boldsymbol{n}] f[\boldsymbol{k}-\boldsymbol{n}] \overset{\boldsymbol{z}}{\longleftrightarrow} G(\boldsymbol{z}) = H(\boldsymbol{z})F(\boldsymbol{z})$$

- Impulse response $h[\boldsymbol{k}]$

- Transfer function $H(z_1, z_2) = \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} h[k_1, k_2] z_1^{-k_1} z_2^{-k_2}$

- Frequency response $H(\mathrm{e}^{\mathrm{j}\omega_1}, \mathrm{e}^{\mathrm{j}\omega_2}) = \sum_{k_1\in\mathbb{Z}} \sum_{k_2\in\mathbb{Z}} h[k_1, k_2] \mathrm{e}^{-\mathrm{j}\omega_1 k_1} \mathrm{e}^{-\mathrm{j}\omega_2 k_2}$

  Eigenfunction property: $(h[\boldsymbol{k}] * \mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{k}}) = H(\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}})\mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{k}}$

Filtering Examples: Revisited

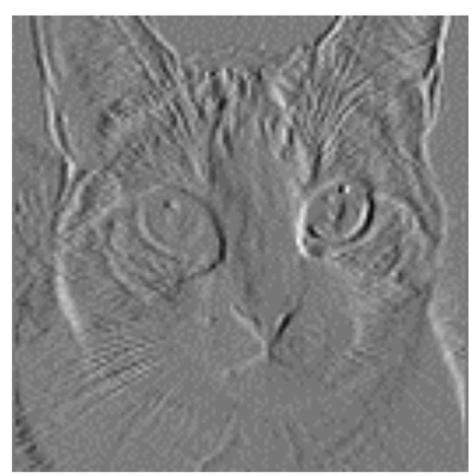
Mask: $\boldsymbol{w}$

– Local (3×3)-average

$$\boldsymbol{w}_{\text{ave}} = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \boxed{1} & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

– Horizontal-edge enhancement

$$\boldsymbol{w}_{\text{hor}} = \begin{bmatrix} -1 & -2 & -1 \\ 0 & \boxed{0} & 0 \\ 1 & 2 & 1 \end{bmatrix}$$



– Vertical-edge enhancement

$$\boldsymbol{w}_{\text{vert}} = \begin{bmatrix} -1 & 0 & 1 \\ -2 & \boxed{0} & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Local Average

- Mask:

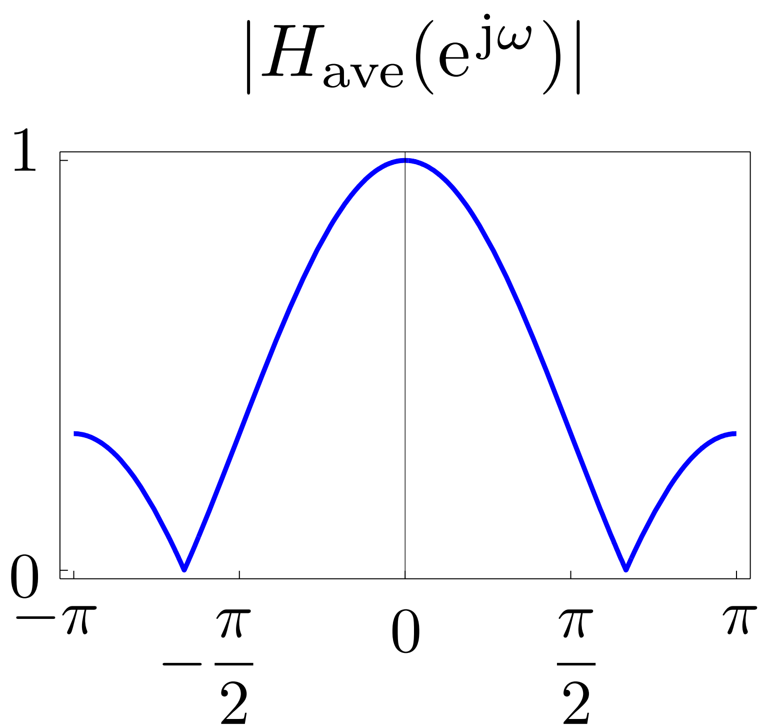
$$\boldsymbol{w}_{\mathrm{ave}} = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \boxed{1} & 1 \\ 1 & 1 & 1 \end{bmatrix} \Longrightarrow \boldsymbol{h}_{\mathrm{ave}} = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \boxed{1} & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Transfer function: $H(z_1, z_2) = \frac{1}{3}(z_1 + 1 + z_1^{-1}) \cdot \frac{1}{3}(z_2 + 1 + z_2^{-1})$

- Frequency response:

$$H(\mathrm{e}^{\mathrm{j}\omega_1}, \mathrm{e}^{\mathrm{j}\omega_2}) = \left(\frac{1 + 2\cos\omega_1}{3}\right)\left(\frac{1 + 2\cos\omega_2}{3}\right)$$

$$= H_{\mathrm{ave}}(\mathrm{e}^{\mathrm{j}\omega_1}) H_{\mathrm{ave}}(\mathrm{e}^{\mathrm{j}\omega_2})$$



low-pass behavior

Vertical-Edge Enhancement

- Mask:

$$\boldsymbol{w}_{\text{vert}} = \begin{bmatrix} -1 & 0 & 1 \\ -2 & \boxed{0} & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{h}_{\text{vert}} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & \boxed{0} & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

"correlation" "convolution"

- Transfer function:

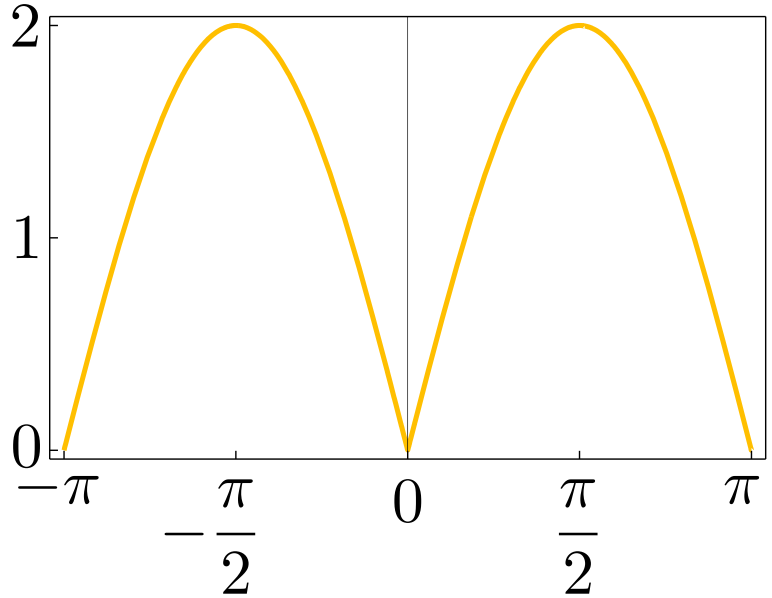
$$H(z_1, z_2) = (z_1 - z_1^{-1})(z_2 + 2 + z_2^{-1})$$

- Frequency response:

$$H(e^{j\omega_1}, e^{j\omega_2}) = (j2\sin\omega_1)(2 + 2\cos\omega_2)$$

$$= H_1(e^{j\omega_1}) H_{\text{low}}(e^{j\omega_2})$$

$|H_1(e^{j\omega})|$



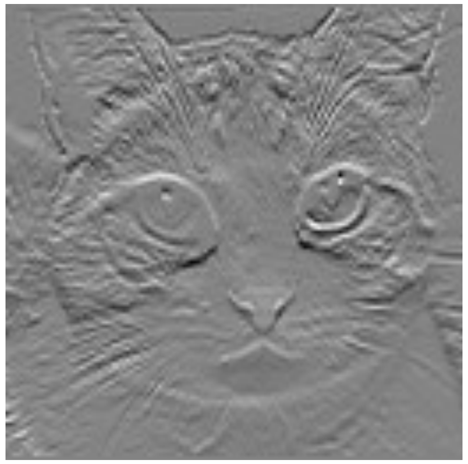
band-pass behavior

Horizontal-edge enhancement is just the "transpose"

Filtering Examples: Separability

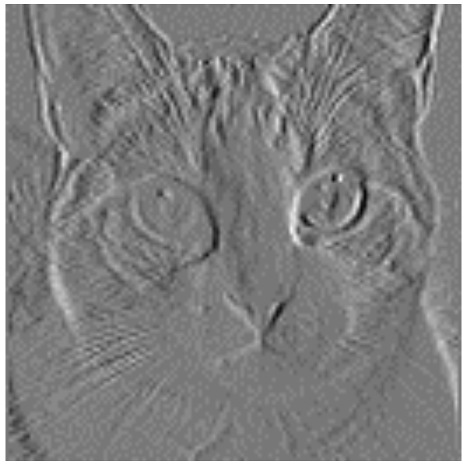
– Local (3×3)-average

$$\boldsymbol{h}_{\text{ave}} = \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \boxed{1} & 1 \\ 1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$



– Horizontal-edge enhancement

$$\boldsymbol{h}_{\text{hor}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & \boxed{0} & 0 \\ -1 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$



– Vertical-edge enhancement

$$\boldsymbol{h}_{\text{vert}} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & \boxed{0} & -2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$