Image Reconstruction

Noise-Free Scenario

- Measurement model:

$$\underset{\mathbb{R}^M}{y} = \underset{\mathbb{R}^{M\times N}}{H}\,\underset{\mathbb{R}^N}{x}$$

Typical regime

$M \ll N$

$[\qquad\quad]$

Q: Given y, how do we recover x?

- Easy case: H is square & invertible

$$\Rightarrow \quad x = H^{-1} y$$

- Practical case: H is not square or not invertible.

Q: How do we know a solution exists?

write $H = \begin{bmatrix} | & & | \\ h_{\bullet 1} & \cdots & h_{\bullet N} \\ | & & | \end{bmatrix}$

Exer: What is Hx in terms of $\{h_n\}_{n=1}^{N}$?

$$Hx = \begin{bmatrix} | & & | \\ h_{\bullet 1} & \cdots & h_{\bullet N} \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

$$= \sum_{n=1}^{N} x_n h_{\bullet n}$$

R(H) "range of H"

Obs: $Hx \in \underbrace{\text{span}(\text{col}(H))}_{R(H)} = S_H$

$\Rightarrow$ a solution exists iff $y \in S_H$

Alternatively, write

$$H = \begin{bmatrix} — & h_{1\bullet}^T & — \\ & \vdots & \\ — & h_{M\bullet}^T & — \end{bmatrix}$$

Then

$$Hx = \begin{bmatrix} h_{1\bullet}^T x \\ \vdots \\ h_{M\bullet}^T x \end{bmatrix}$$

If $\operatorname{rank}(H) = M$ and H has $N > M$ nonzero columns, then there exists $x_0 \in \mathbb{R}^N$, $x_0 \neq 0$, such that $H x_0 = 0$, i.e.,

$$h_{m\cdot}^T x_0 = 0, \quad m = 1, \dots, M.$$

x_0 is <u>orthogonal</u> to the rows of H.

The <u>null</u> <u>space</u> of H is

$$\operatorname{null}(H) = \{ x \in \mathbb{R}^N : Hx = 0 \}$$

In inverse problems, any $v \in \operatorname{null}(H)$ is called a "ghost".

<u>Q:</u> How big can a null space be?

If $x_0 \in null(H)$, then $\alpha x_0 \in null(H)$.

- $null(H) = \{0\}$ one element
- $null(H)$ is infinitely large.

Obs: if $\hat{x}$ is a solution to $y = Hx$, then so is

$$\tilde{x} = \hat{x} + v, \quad v \in null(H)$$

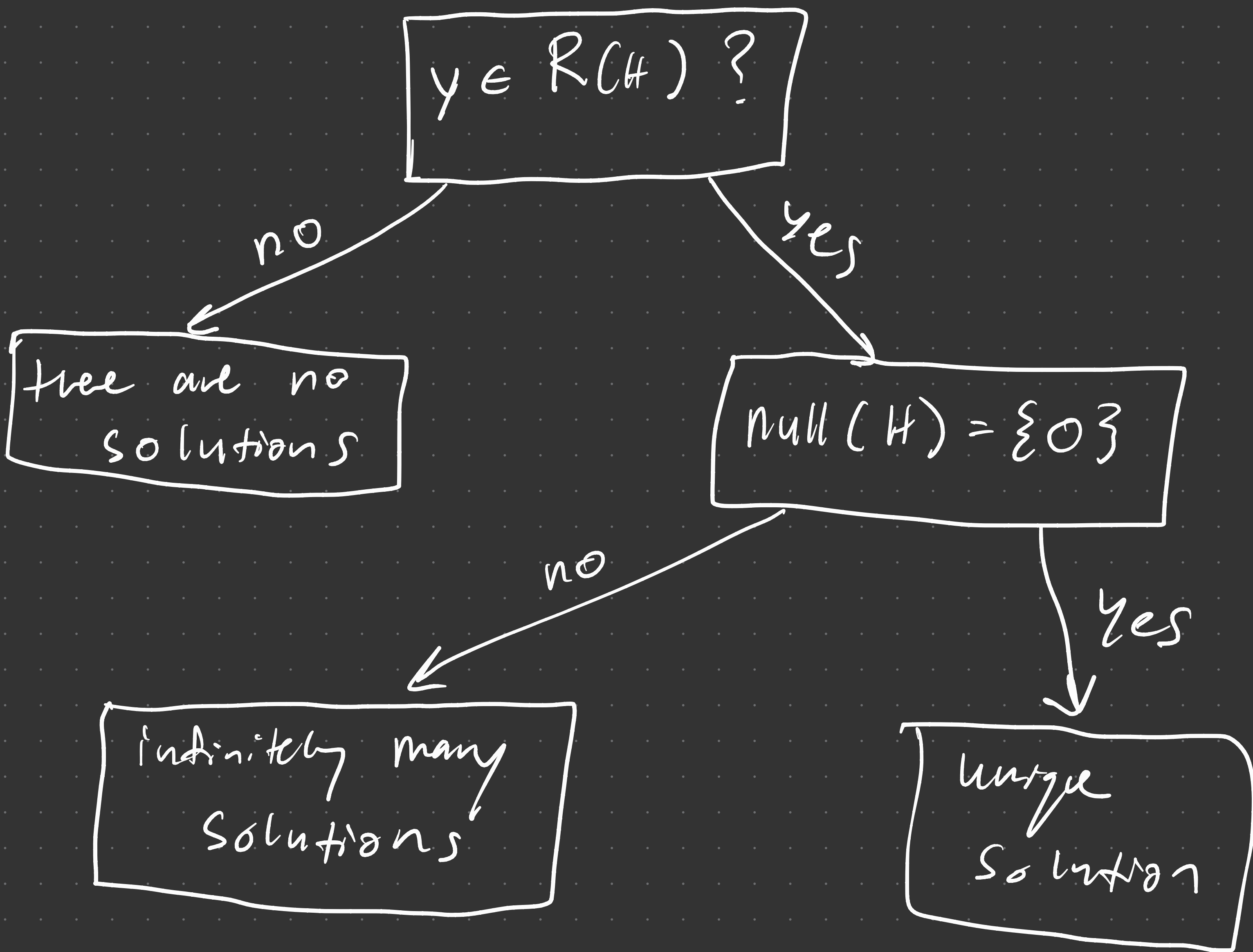
since

$$H\tilde{x} = H(\hat{x} + v) = H\hat{x} + \underbrace{\cancel{Hv}}_{0} = H\hat{x}$$

If $y = Hx$, then there are two possibilities:

1. $null(H) = \{0\} \Rightarrow$ unique solution
2. $|null(H)| > 1 \Rightarrow$ infinite # of solutions

Flowchart of existence of solutions to $y = Hx$



<u>Remark:</u> The favorable scenario (yes-yes branch) (almost) never happens in practice

<u>Q:</u> What do we do if there are no solutions? Can we find x such that $y \approx Hx$?

Least-Squares Solution

Solve

$$\min_{x \in \mathbb{R}^N} \|y - Hx\|_2^2$$

i.e., minimize $\sum_{m=1}^{M} (\tilde{y}_m - y_m)^2$, where $\tilde{y} = Hx$.

Vector Calculus Review

$J : \mathbb{R}^N \to \mathbb{R}$, $J(v)$

$$\frac{\partial J(v)}{\partial v} = \begin{bmatrix} \partial J(v)/\partial v_1 \\ \vdots \\ \partial J(v)/\partial v_N \end{bmatrix} = \nabla J(v)$$

Necessary condition for optimality:

$$\frac{\partial J(v)}{\partial v} = 0$$

(Also sufficient if $J(v)$ is cvx in v)

Useful identities:

$$\frac{\partial}{\partial v}(a^T v) = \frac{\partial}{\partial v}(v^T a) = a$$

$$\frac{\partial}{\partial v}(v^T A v) = (A + A^T)\, v$$

if A is symmetric:

$$\frac{\partial}{\partial v}(v^T A v) = 2 A v$$

Singular Value Decomposition (SVD)

Let $H \in \mathbb{R}^{M \times N}$ and $\text{rank}(H) = r$.

Then, there exists orthonormal vectors

$u_1, \dots, u_M \in \mathbb{R}^M$, $U = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_M \\ | & & | \end{bmatrix} \in \mathbb{R}^{M \times M}$

$v_1, \dots, v_N \in \mathbb{R}^N$, $V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_N \\ | & & | \end{bmatrix} \in \mathbb{R}^{N \times N}$

Such that

$$H = U \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_N \\ & 0 & \end{bmatrix} V^T, \quad \text{when } M \geq N$$

or

$$H = U \begin{bmatrix} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_M & \end{bmatrix} V^T, \quad \text{when } M \leq N$$

with $\sigma_j = 0$ for all $j > r$.

Alternatively:

$$H = \begin{array}{cc} & \begin{array}{cc} r & M-r \end{array} \\ M & \begin{bmatrix} U_1 & U_2 \end{bmatrix} \end{array} \underbrace{\begin{array}{cc} & \begin{array}{cc} r & N-r \end{array} \\ \begin{array}{c} r \\ M-r \end{array} & \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \end{array}}_{\Sigma} \begin{array}{cc} N & \\ \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} & \begin{array}{c} r \\ N-r \end{array} \end{array}$$

whee

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix} = \mathrm{diag}(\sigma_1, \dots, \sigma_r)$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$$

Compact SVD:

$$H = U_1 \Sigma_1 V_1^T$$

$$= \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 v_1^T \\ \vdots \\ \sigma_r v_r^T \end{bmatrix}$$

$$= \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

"Sum of rank 1 matrices"

Theorem (Eckart-Young-Mirsky):

Let $H \in \mathbb{R}^{M \times N}$. Then,

$$\min_A \|H - A\|_F^2 \quad \text{s.t.} \quad \operatorname{rank}(A) \leq r$$

$\|A\|_F^2 = \sum_{i,j} a_{ij}^2$

is achieved by

$$\hat{A} = \sum_{i=1}^{r} \sigma_i u_i v_i^T,$$

where $H = U \Sigma V^T$ is the SVD of H.

Back to Least-Squares for Inverse Problems

$$\min_{x \in \mathbb{R}^N} \underbrace{\|y - Hx\|_2^2}_{J(x)}, \quad H \in \mathbb{R}^{M \times N}, \quad y \in \mathbb{R}^M$$

Q: When is this minimized?

A: $\frac{\partial J(x)}{\partial x} = 0$. ($J$ is convex)

$$J(x) = (y - Hx)^T (y - Hx)$$

$$= y^T y - y^T H x - x^T H^T y + x^T H^T H x$$

$$= y^T y - 2 y^T H x \quad + x^T H^T H x$$

$$0 = \frac{\partial J(x)}{\partial x} = -2 H^T y + 2 H^T H x$$

$$\Rightarrow \quad H^T y = H^T H x$$

<u>Q:</u> Does this always have a solution?

<u>A:</u> Yes, because $H^T y \in \text{span}(\text{col}(H^T))$

$= \text{span}(\text{row}(H))$

But, the solution may not be unique.

(more on this later)

<u>Obs:</u> if $H^T H$ is invertible

$$x = (H^T H)^{-1} H^T y$$

Iterative Solution to Least Squares

Want to solve:

$$H^T y = H^T H x$$

Let x_0 be an initial guess. Then

$$e = H^T H x_0 - H^T y \in \mathbb{R}^N$$

is the "error" of the guess. We can try to reduce the error by considering

$$x_1 = x_0 - \mu e, \quad \text{for some small } \mu > 0.$$

This process can be repeated

$$x_{k+1} = x_k - \mu H^T(H x_k - y), \quad k = 0, 1, 2, \ldots$$

Observe that if

$$x_k \longrightarrow \tilde{x} \quad \text{as} \quad k \to \infty,$$

then

$$x_{k+1} - x_k \longrightarrow 0$$

$$\Rightarrow \quad H^T H x_k - H^T y \rightarrow 0$$

$$\Rightarrow \quad \tilde{x} \text{ is a solution to}$$

$$\min_{x \in \mathbb{R}^N} \|y - Hx\|_2^2 .$$

<u>Theorem</u>: Let $x_0 \in \mathbb{R}^N$ be arbitrary and let

$$x_{k+1} = x_k - \mu H^T (H x_k - y), \quad k=0,1,\ldots$$

Then, x_k converges to a solution of

$$\min_{x \in \mathbb{R}^N} \|y - Hx\|_2^2$$

<u>if and only if</u> $0 < \mu < \dfrac{2}{\sigma_1^2(H)}$.

Proof (Simplified):

Additionally suppose that $H^T H$ is invertible.
Then, the solution to

$$H^T y = H^T H x$$

is $x^* = (H^T H)^{-1} H^T y$ (unique).

$$x_{k+1} = x_k - \mu H^T (H x_k - y)$$

$$= x_k - \mu (H^T H x_k - H^T y)$$

$$= x_k - \mu (H^T H) (x_k - \underbrace{(H^T H)^{-1} H^T y}_{x^*})$$

$$x_{k+1} - x^* = (x_k - x^*) - \mu (H^T H) (x_k - x^*)$$

Define $z_k = x_k - x^*$

$$z_{k+1} = z_k - \mu H^T H z_k$$

$$= (I - \mu H^T H) z_k$$

Let $H = U \Sigma V^T$ be the SVD of H

and note that $V^T V = I_N$, $U^T U = I_M$

$$z_{k+1} = (V I V^T - \mu V \Sigma^T \underbrace{U^T U}_{I} \Sigma V^T) z_k$$

$$= V(I - \mu \Sigma^T \Sigma) V^T z_k$$

$$V^T z_{k+1} = (I - \mu \Sigma^T \Sigma) V^T z_k$$

Define $\theta_k = V^T z_k \in \mathbb{R}^N$

$$\theta_{k+1} = (I - \mu \Sigma^T \Sigma) \theta_k$$

Consider the nth component of θ_k

$$\theta_{k+1,n} = (1 - \mu \sigma_n^2) \theta_{k,n} \in \mathbb{R}$$

When does this iteration converge?

It converges if and only if

$$|1 - \mu \sigma_n^2| < 1$$

for all $n = 1, \dots, N$

Two cases:

① $\underline{\mu \sigma_n^2 < 1}$

$$1 - \mu \sigma_n^2 < 1$$

$$\mu \sigma_n^2 > 0$$

$$\mu > 0$$

② $\underline{\mu \sigma_n^2 \geq 1}$

$$-1 + \mu \sigma_n^2 < 1$$

$$\mu \sigma_n^2 < 2$$

$$\mu < \frac{2}{\sigma_n^2}$$

This holds for all $n = 1, \dots, N$ and so

$$\mu < \frac{2}{\sigma_1^2} \quad \text{(sorted sing. values)}$$

$\blacksquare$

Regularization

A common approach to deal with non uniqueness of solutions as well as noisy data is to regularize

$$\min_{x \in \mathbb{R}^N} \underbrace{\|y - Hx\|_2^2 + \lambda \|x\|_2^2}_{J(x)}$$

Exercise: Find an expression for the optimal solution.

$$0 = \frac{\partial J(x)}{\partial x} = -2H^T y + 2H^T H x + 2\lambda I x$$

$$\Rightarrow H^T y = \underbrace{(H^T H + \lambda I)}_{\text{always invertible}} x$$

$$\Rightarrow x = (H^T H + \lambda I)^{-1} H^T y$$

unique solution!

Problem Variants

$$\min_{x \in \mathbb{R}^N} \|y - Hx\|_2^2 + \lambda \|Lx\|_2^2, \quad \lambda > 0,$$

where $L \in \mathbb{R}^{K \times N}$ is some fixed matrix.

Exercise:

1. Find a system of equations a solution must satisfy
2. Under what conditions on L is the solution unique?

Observe that if we define

$$\tilde{H} = \begin{bmatrix} H \\ \sqrt{\lambda} L \end{bmatrix}, \quad \tilde{y} = \begin{bmatrix} y \\ 0 \end{bmatrix},$$

then

$$\|\tilde{y} - \tilde{H}x\|_2^2 = \left\| \begin{bmatrix} y \\ 0 \end{bmatrix} - \begin{bmatrix} H \\ \sqrt{\lambda} L \end{bmatrix} x \right\|_2^2$$

$$= \left\| \begin{bmatrix} y - Hx \\ -\sqrt{\lambda} Lx \end{bmatrix} \right\|_2^2$$

$$= \|y - Hx\|_2^2 + \lambda \|Lx\|_2^2$$

Then, we already know a solution must solve

$$\tilde{H}^T \tilde{H} x = \tilde{H}^T \tilde{y}$$

$$\Longleftrightarrow \begin{bmatrix} H^T & \sqrt{\lambda} L^T \end{bmatrix} \begin{bmatrix} H \\ \sqrt{\lambda} L \end{bmatrix} x = \begin{bmatrix} H^T & \sqrt{\lambda} L^T \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix}$$

$$\Longleftrightarrow \boxed{(H^T H + \lambda L^T L) x = H^T y}$$

Solution is unique if and only if

$$(H^T H + \lambda L^T L)$$

is invertible. This is guaranteed if $L^T L$ is invertible $\iff \mathrm{null}(L) = \{0\}$.

In which case, the solution is

$$x^* = (H^T H + \lambda L^T L)^{-1} H^T y$$