ECE 172A: Introduction to Image Processing
Image Reconstruction: Part I

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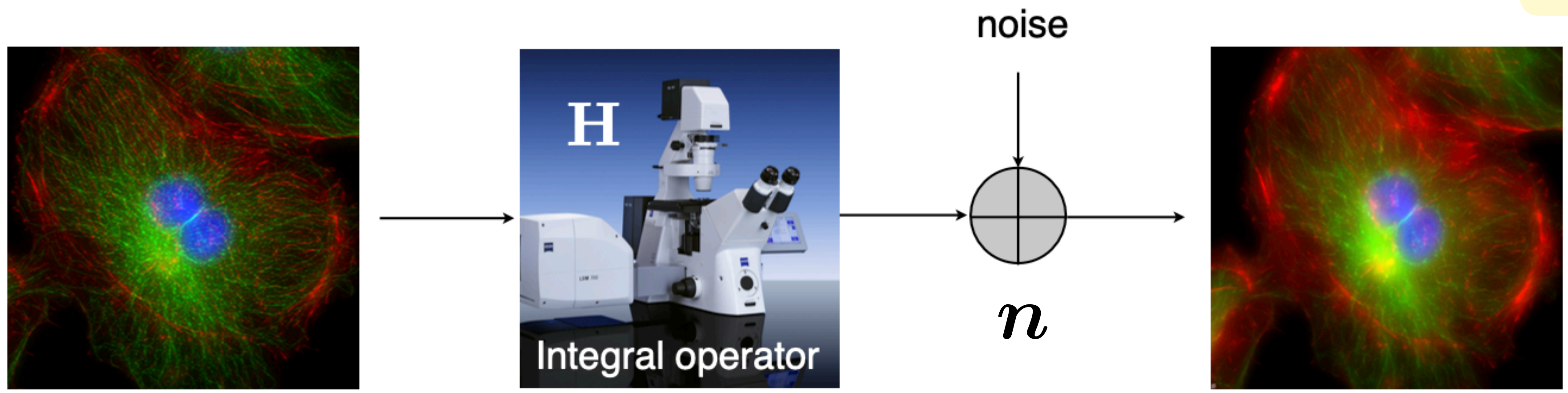
Outline

- Image reconstruction as an inverse problem
- Imaging operators and modalities
- Discretization of inverse problems

Image Reconstruction as an Inverse Problem

$$\boldsymbol{y} = \mathbf{H}\boldsymbol{x} + \boldsymbol{n}$$

- Linear forward model



$\boldsymbol{x}$ object

$\boldsymbol{y}$ data

Given $\boldsymbol{y}$, reconstruct $\boldsymbol{x}$

"invert $\mathbf{H}$"

Why is this problem hard?

- noise amplification
- difficult to invert $\mathbf{H}$ (large or non-square)
- all interesting inverse problems are **ill-posed**

- Backprojection (poor man's solution): $\widehat{\boldsymbol{x}} = \mathbf{H}^\top \boldsymbol{y}$

What are some examples of inverse problems?

Imaging Operators and Modalities

- Fourier Transform
- Windowing
- Convolution
- Radon Transform

Mathematical Formulation of Forward Models

- Unknown object: $s(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^2$ (or more generally $\boldsymbol{x} \in \mathbb{R}^d$)

  defined in the analog domain

  $$s \in L^2(\mathbb{R}^2)$$ (finite-energy objects)

- Imaging operator/forward model: $\mathrm{H}\{s\} = \boldsymbol{y} \in \mathbb{R}^M$

  from analog to discrete

  $$\mathrm{H} : L^2(\mathbb{R}^2) \to \mathbb{R}^M$$

- Boundedness (BIBO) and linearity assumption:

  $\|\mathrm{H}\{s\}\|_2 \leq C\|s\|_{L^2}$

  $\mathrm{H}\{\alpha_1 s_1 + \alpha_2 s_2\} = \alpha_1 \mathrm{H}\{s_1\} + \alpha_2 \mathrm{H}\{s_2\}$, for all $s_1, s_2 \in L^2(\mathbb{R}^2), \alpha_1, \alpha_2 \in \mathbb{R}$

  $$\Rightarrow y_m = \langle \eta_m, s \rangle = \int_{\mathbb{R}^2} \eta_m(\boldsymbol{x}) s(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$ *(Riesz representation theorem)*

$\{\eta_m\}_{m=1}^M$ are called the **analysis functions**

Basic Operator: Fourier Transform

$$\mathcal{F} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$

$$\hat{s}(\boldsymbol{\omega}) = \mathcal{F}\{s\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} s(\boldsymbol{x}) \mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}} \, \mathrm{d}\boldsymbol{x}$$

Reconstruction formula (inverse Fourier transform)

$$s(\boldsymbol{x}) = \mathcal{F}^{-1}\{\hat{s}\}(\boldsymbol{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{s}(\boldsymbol{\omega}) \mathrm{e}^{\mathrm{j}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{x}} \, \mathrm{d}\boldsymbol{\omega}$$

Exercise: What would be the analysis functions of the forward model that samples the Fourier transform at $\{\boldsymbol{\omega}_m\}_{m=1}^M$?

Basic Operator: Windowing

$$\mathrm{W}: L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$

$$\mathrm{W}\{s\}(\boldsymbol{x}) = w(\boldsymbol{x})s(\boldsymbol{x})$$

w is a positive and bounded window function:

$$w(\boldsymbol{x}) \geq 0 \text{ and } w(\boldsymbol{x}) \leq C \text{ for all } \boldsymbol{x} \in \mathbb{R}^2$$

- Special case: Modulation

  $w(\boldsymbol{x}) = \mathrm{e}^{\mathrm{j}\boldsymbol{\omega}_0^{\mathsf{T}}\boldsymbol{x}}$ Structured Illumination Microscopy (SIM)

Magnetic Resonance Imaging

- Simplified forward model for MRI

$$\hat{s}(\boldsymbol{\omega}_m) = \int_{\mathbb{R}^2} s(\boldsymbol{x}) \mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}_m^{\mathsf{T}}\boldsymbol{x}} \, \mathrm{d}\boldsymbol{x}$$

(sampling of the Fourier transform)

- More-realistic forward model for MRI

$$\hat{s}_w(\boldsymbol{\omega}_m) = \int_{\mathbb{R}^2} w(\boldsymbol{x}) s(\boldsymbol{x}) \mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}_m^{\mathsf{T}}\boldsymbol{x}} \, \mathrm{d}\boldsymbol{x}$$

(sampling of the short-time (short-space?) Fourier transform)

(window models the effect of the coil)

Basic Operator: Convolution

$$\mathrm{H} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$$

$$\mathrm{H}\{s\}(\boldsymbol{x}) = (h * s)(\boldsymbol{x}) = \int_{\mathbb{R}^2} h(\boldsymbol{x} - \boldsymbol{y}) s(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}$$

Impulse response: $h = \mathrm{H}\{\delta\}$

- Convolution as frequency-domain product: $(h * s)(\boldsymbol{x}) \xleftrightarrow{\mathcal{F}} \hat{h}(\boldsymbol{\omega})\hat{s}(\boldsymbol{\omega})$

Exercise: What would be the analysis functions of the forward model that samples the convolution at $\{\boldsymbol{x}_m\}_{m=1}^{M}$?

Basic Operator: Radon Transform

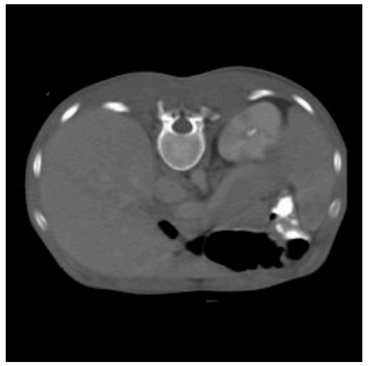
- Radon transform (line integrals)

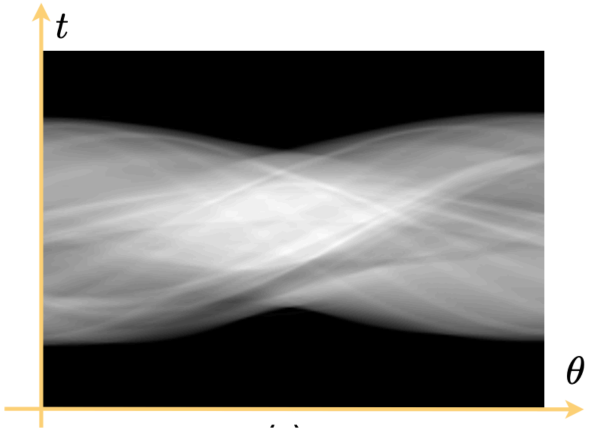
Notation: $\boldsymbol{\theta} = (\cos\theta, \sin\theta) \in \mathbb{R}^2$

A line in $\mathbb{R}^2$ can be represented by all $\boldsymbol{x} \in \mathbb{R}^2$ such that

$$\boldsymbol{\theta}^{\mathsf{T}}\boldsymbol{x} = t \qquad \Leftrightarrow \qquad x\cos\theta + y\sin\theta = t$$

$$\mathscr{R}_\theta\{f\}(t) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\delta(\boldsymbol{\theta}^{\mathsf{T}}\boldsymbol{x} - t)\,\mathrm{d}\boldsymbol{x}$$





sinogram

Exercise: What would be the analysis functions of the forward model that samples the Radon transform at $\{(\theta_m, t_m)\}_{m=1}^{M}$?

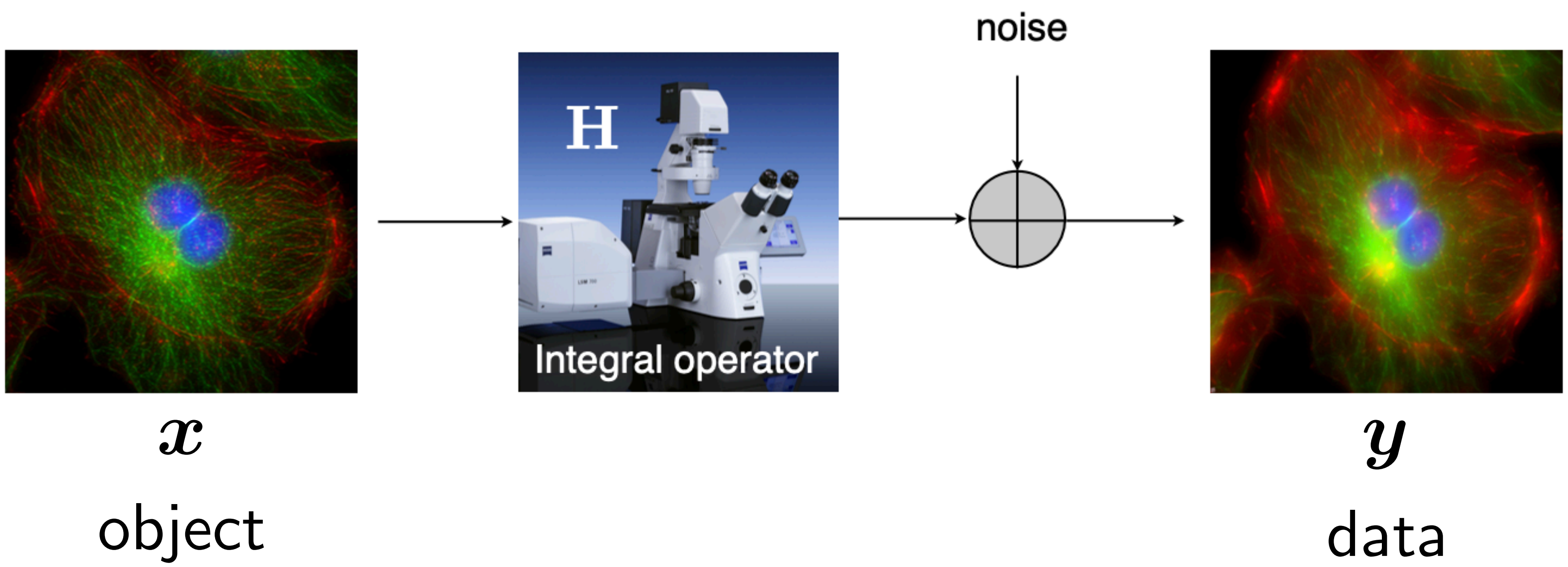
Panorama of Imaging Modalities

Modality	Radiation	Forward model	Variations
2D or 3D tomography	coherent x-ray	$y_i = \mathrm{R}_{\boldsymbol{\theta}_i} x$	parallel, cone beam, spiral sampling
3D deconvolution microscopy	fluorescence	$y = \mathrm{H}x$	brightfield, confocal, light sheet
structured illumination microscopy (SIM)	fluorescence	$y_i = \mathrm{H}\mathrm{W}_i x$ H: PSF of microscope W_i: illumination pattern	full 3D reconstruction, non-sinusoidal patterns
Positron Emission Tomography (PET)	gamma rays	$y_i = \mathrm{H}_{\boldsymbol{\theta}_i} x$	list mode with time-of-flight
Magnetic resonance imaging (MRI)	radio frequency	$y = \mathrm{F}x$	uniform or non-uniform sampling in k space
Cardiac MRI (parallel, non-uniform)	radio frequency	$y_{t,i} = \mathrm{F}_t \mathrm{W}_i x$ W_i: coil sensitivity	gated or not, retrospective registration
Optical diffraction tomography	coherent light	$y_i = \mathrm{W}_i \mathrm{F}_i x$	with holography or grating interferometry

Discretization

- Discretization: How to turn the problem into linear algebra
- Examples
  - Diffraction-limited convolution (Fluorescence microscopy)
  - MRI
  - CT

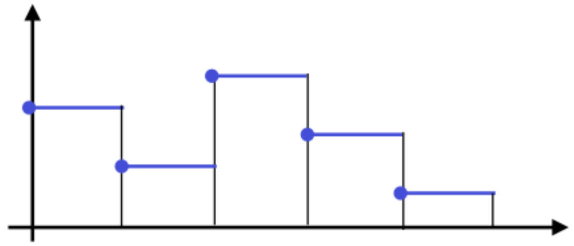
$$\boldsymbol{y} = \mathbf{H}\boldsymbol{x} + \boldsymbol{n}$$



Discretization: Finite-Dimensional Formulation

Selection of an appropriate basis of functions $\beta_{\boldsymbol{k}} : \mathbb{R}^2 \to \mathbb{R}$ with $\boldsymbol{k} \in \Omega \subset \mathbb{Z}^2$ and $|\Omega| = N$.

Assume, *a priori*, that $$s(\boldsymbol{r}) = \sum_{\boldsymbol{k}\in\Omega} s[\boldsymbol{k}]\beta_{\boldsymbol{k}}(\boldsymbol{r})$$



Create a vector $\boldsymbol{x} = (s[\boldsymbol{k}])_{\boldsymbol{k}\in\Omega} \in \mathbb{R}^N$

- Measurement model (image formation):

$$y_m = \int_{\mathbb{R}^2} s(\boldsymbol{r})\eta_m(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} + n[m] = \langle s, \eta_m\rangle + n[m],\ m = 1, \dots, M$$

η_m: mth detector (analysis function)

$n[m]$: additive noise

$$[\mathbf{H}]_{m,\boldsymbol{k}} = \langle \eta_m, \beta_{\boldsymbol{k}}\rangle = \int_{\mathbb{R}^2} \eta_m(\boldsymbol{r})\beta_{\boldsymbol{k}}(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}$$

Reshape into $\mathbf{H} \in \mathbb{R}^{M\times N}$:
$\boldsymbol{y} = \mathbf{H}\boldsymbol{x} + \boldsymbol{n}$

We have now reduced the problem to a linear algebra problem (given $\boldsymbol{x}$, we can synthesize $s(\boldsymbol{r})$)

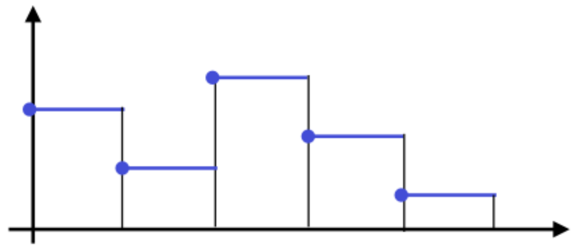
Examples of Basis Functions

Shift-invariant representation: $\beta_{\boldsymbol{k}}(\boldsymbol{x}) = \beta(\boldsymbol{x} - \boldsymbol{k})$

Separable generator: $\beta(\boldsymbol{x}) = \beta(x)\beta(y)$

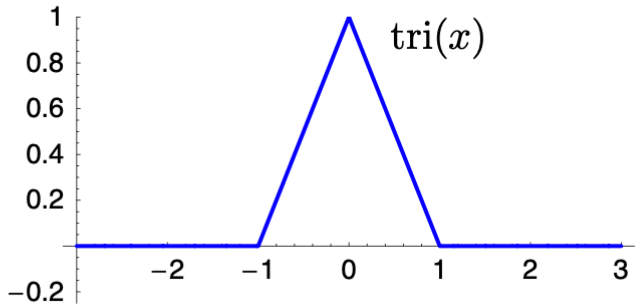
- Pixel basis:

$$\beta(x) = \mathrm{rect}(x)$$



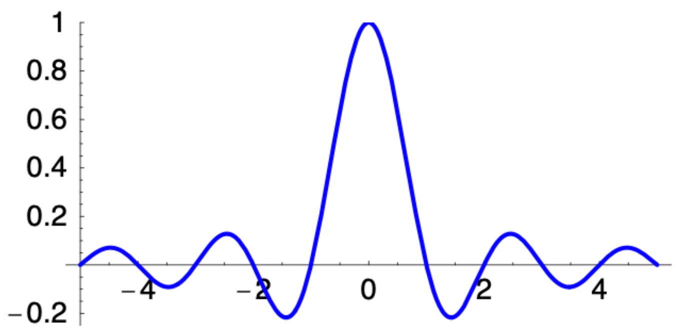
- Bilinear basis:

$$\beta(x) = (\mathrm{rect} * \mathrm{rect})(x) = \mathrm{tri}(x)$$



- Bandlimited basis:

$$\beta(x) = \mathrm{sinc}(x)$$



Example 1: Diffraction-Limited Convolution

Hypothesis: $\hat{h}_{2\mathrm{D}}(\boldsymbol{\omega}) = 0$ for $\|\boldsymbol{\omega}\| \geq \omega_0$ (diffraction-limited)

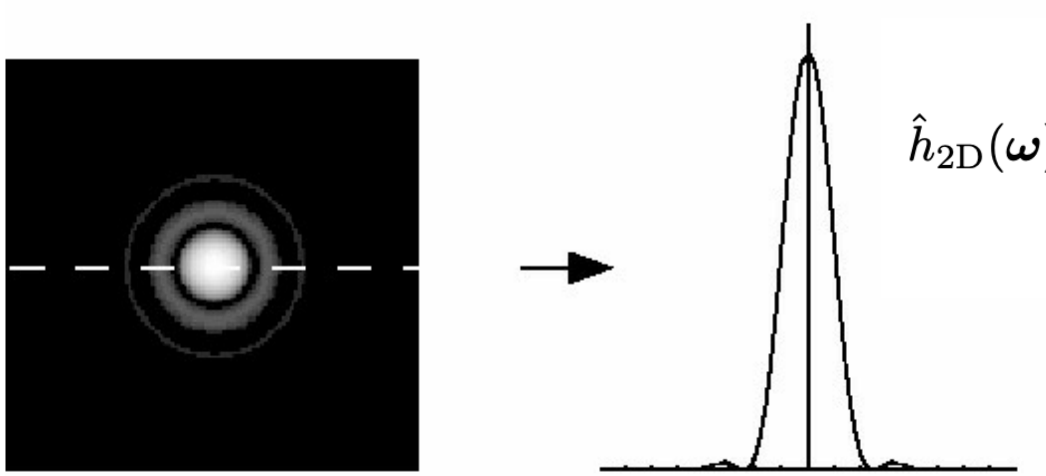
- Discretization

  $\omega_0 \leq \pi$ and representation in (separable) sinc basis $\beta_{\boldsymbol{k}}(\boldsymbol{x}) = \mathrm{sinc}(\boldsymbol{x} - \boldsymbol{k})$

  Analysis functions: $\eta_{\boldsymbol{m}}(\boldsymbol{x}) = h_{2\mathrm{D}}(\boldsymbol{x} - \boldsymbol{m})$

$$[\mathbf{H}]_{\boldsymbol{m},\boldsymbol{k}} = \langle \eta_{\boldsymbol{m}}(\boldsymbol{x}), \mathrm{sinc}(\boldsymbol{x} - \boldsymbol{k}) \rangle$$

$$= h_{2\mathrm{D}}(\boldsymbol{m} - \boldsymbol{k})$$



Airy Disk → Radial Profile

$$\hat{h}_{2\mathrm{D}}(\boldsymbol{\omega}) = \begin{cases} \frac{2}{\pi}\left(\arccos\left(\frac{\|\boldsymbol{\omega}\|}{\omega_0}\right) - \frac{\|\boldsymbol{\omega}\|}{\omega_0}\sqrt{1 - \left(\frac{\|\boldsymbol{\omega}\|}{\omega_0}\right)^2}\right), & \text{for } 0 \leq \|\boldsymbol{\omega}\| < \omega_0 \\ 0, & \text{otherwise} \end{cases}$$

Example 2: Magnetic Resonance Imaging (MRI)

- Simplified forward model for MRI

$$\hat{s}(\boldsymbol{\omega}_m) = \int_{\mathbb{R}^2} s(\boldsymbol{x})\mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}_m^{\mathsf{T}}\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}$$

(sampling of the Fourier transform)

$$\eta_m(\boldsymbol{x}) = \mathrm{e}^{\mathrm{j}\boldsymbol{\omega}_m^{\mathsf{T}}\boldsymbol{x}}$$

suppose that $\|\boldsymbol{\omega}_m\|_2 < \pi$

- Discretization in separable sinc basis

$$[\mathbf{H}]_{m,\boldsymbol{k}} = \langle \eta_m(\boldsymbol{x}), \mathrm{sinc}(\boldsymbol{x} - \boldsymbol{k})\rangle$$
$$= \langle \mathrm{e}^{\mathrm{j}\boldsymbol{\omega}_m^{\mathsf{T}}\boldsymbol{x}}, \mathrm{sinc}(\boldsymbol{x} - \boldsymbol{k})\rangle = \mathrm{e}^{-\mathrm{j}\boldsymbol{\omega}_m^{\mathsf{T}}\boldsymbol{k}}$$

Example 3: Computed Tomography (CT)

- Radon transform (line integrals)

  Notation: $\boldsymbol{\theta} = (\cos\theta, \sin\theta) \in \mathbb{R}^2$

$$\mathscr{R}_\theta\{f\}(t) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\delta(\boldsymbol{\theta}^\mathsf{T}\boldsymbol{x} - t)\,\mathrm{d}\boldsymbol{x}$$

Recall the Radon transform of a separable function $\varphi(\boldsymbol{x}) = \varphi_1(x)\varphi_2(y)$:

$$\mathscr{R}_\theta\{\varphi\}(t) = \varphi_\theta(t),$$

where

$$\varphi_\theta(t) = \frac{1}{|\cos\theta|}\varphi_1\left(\frac{t}{\cos\theta}\right) * \frac{1}{|\sin\theta|}\varphi_2\left(\frac{t}{\sin\theta}\right)$$

$$[\mathbf{H}]_{(i,j),\boldsymbol{n}} = \mathscr{R}_{\theta_i}\{\varphi(\boldsymbol{x} - \boldsymbol{n})\}(t_j) = \varphi_{\theta_i}(t_j - \boldsymbol{n}^\mathsf{T}\boldsymbol{\theta}_i)$$

Finite-Dimensional Inverse Problem

- Discretized forward model:

$$\boldsymbol{y} = \mathbf{H}\boldsymbol{x} + \boldsymbol{n}$$

$$\boldsymbol{x} \in \mathbb{R}^N$$

$$\mathbf{H} \in \mathbb{R}^{M \times N}$$

$$\boldsymbol{y} \in \mathbb{R}^M$$

$$\boldsymbol{n} \in \mathbb{R}^M$$

- Inverse problem: How to efficiently recover $\boldsymbol{x}$ from $\boldsymbol{y}$?