Welcome to ECE 251C

Filter Banks and Wavelets
(Multirate Signal Processing)

Course will primarily focus on:

① Filters

② Filter Banks

③ Wavelets

Lectures: Theory

HW: Theory and MATLAB (or python)

Project: Theory OR Applications

Filters / LTI Systems (Review)

Two kinds of filters:

① Finite Impulse Response (FIR)

② Infinite Impulse Response (IIR)

Q: What is the difference?

Q: What is an impulse response?

Linear
Time (Translation)
Invariant

sampled signals $n \in \mathbb{Z}$

$$x[n] \longrightarrow \boxed{\qquad} \longrightarrow y[n]$$

Linear:

$$x_1[n] \longrightarrow \boxed{\qquad} \longrightarrow y_1[n]$$

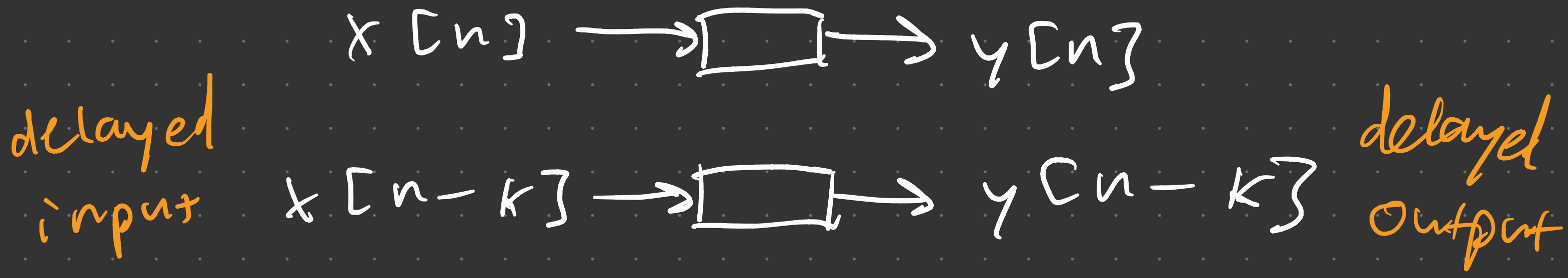
$$x_2[n] \longrightarrow \boxed{\qquad} \longrightarrow y_2[n]$$

$\forall \alpha, \beta \in \mathbb{R}$

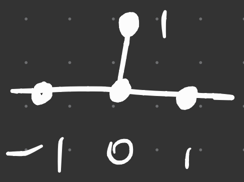
$$\alpha x_1[n] + \beta x_2[n] \longrightarrow \boxed{\qquad} \longrightarrow \alpha y_1[n] + \beta y_2[n]$$

Time/Translation Invariant:

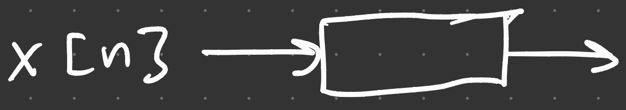
The system does not care when it is turned on.



Impulse Response



"impulse response"



$$y[n] = (h * x)[n]$$

$$= \sum_{k \in \mathbb{Z}} h[k]\, x[n-k]$$

FIR: Support of h is finite

IIR: Support of h is infinite

Support = # of nonzero values

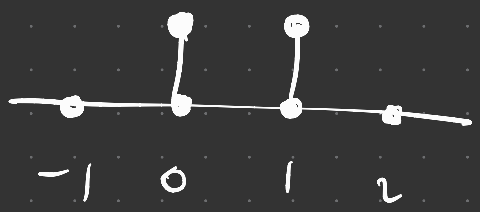
Ex: $$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right)$$

Q: What does this system do?

A:
- Moving average filter
- Smoothing
- Low-pass filter

Impulse response:

$$h[n] = \frac{1}{2}\left(\delta[n] + \delta[n-1]\right)$$



FIR

2-tap filter

Q: How do we know it is low-pass?

A: Look at the frequency response

$$\underline{e^{j\omega n}} \longrightarrow \boxed{} \longrightarrow \sum_{k\in\mathbb{Z}} h[k]\, e^{j\omega(n-k)}$$

$$= e^{j\omega n}\left(\sum_{k\in\mathbb{Z}} h[k]\, e^{-j\omega k}\right)$$

$$= \underline{e^{j\omega n}}\ \underline{H(e^{j\omega})}$$

DTFT / frequency response

Obs: Pure frequencies (complex exponentials) are eigenfunctions of LTI systems.

Convolution Theorem:

$$y[n] = (h * x)[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

Signal-processing notation

$$Y(\omega) = H(\omega)\, X(\omega)$$

reduced / math notation

Z - Transform

$$X(z) = \sum_{n \in \mathbb{Z}} x[n] z^{-n} \quad , \quad z \in \mathbb{C}$$

$$X(e^{j\omega}) = X(z)\Big|_{z = e^{j\omega}}$$

Ex: $h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$

$$H(e^{j\omega}) = \frac{1}{2} + \frac{1}{2} e^{-j\omega}$$

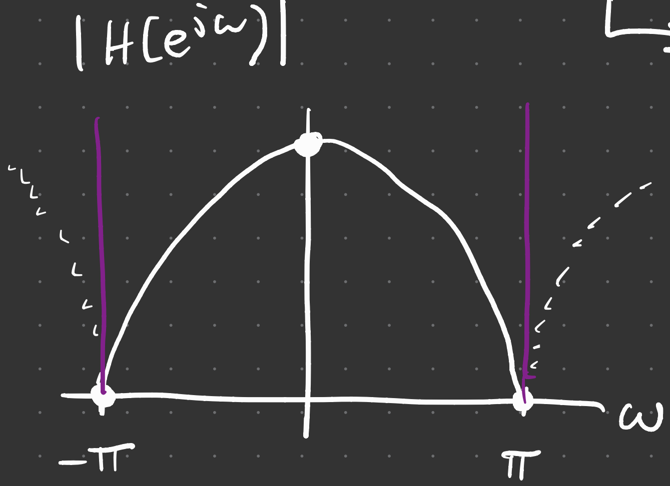
$$= e^{-\frac{j\omega}{2}} \left[\frac{e^{\frac{j\omega}{2}} + e^{-j\frac{\omega}{2}}}{2} \right]$$

$$= e^{-j\frac{\omega}{2}} \cos\left(\frac{\omega}{2}\right)$$

phase magnitude

$$\hookrightarrow = e^{j\Phi(\omega)} \Rightarrow \Phi(\omega) = -\frac{\omega}{2}$$

phase response

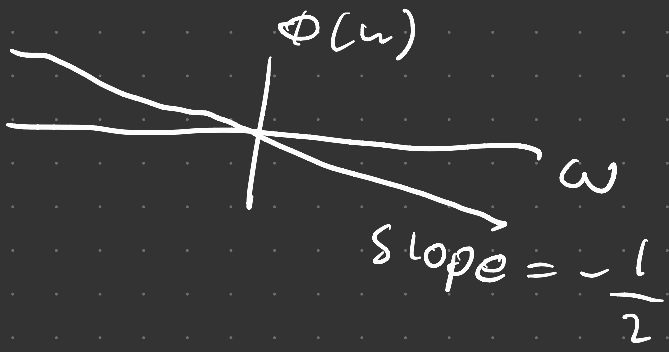


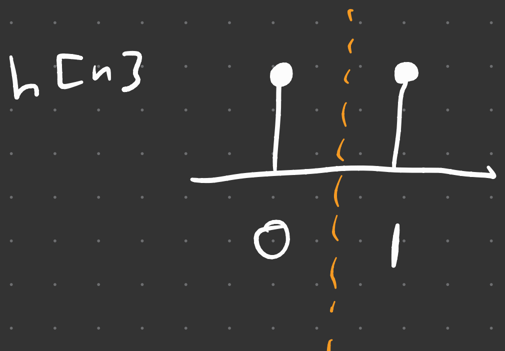
low-pass
(attenuating high frequencies)

Linear-Phase Systems

<u>Defn</u>: Given $H(e^{j\omega}) = e^{j\Phi(\omega)} |H(e^{j\omega})|$, the system is called <u>linear-phase</u> if $\Phi(\omega)$ is linear.

<u>Ex:</u> $\Phi(\omega) = -\frac{\omega}{2}$





$\frac{1}{2}$ = center of symmetry

Symmetric / Antisymmetric filters provide a <u>complete</u> characterization of linear-phase systems.

Type I : odd length / symmetric

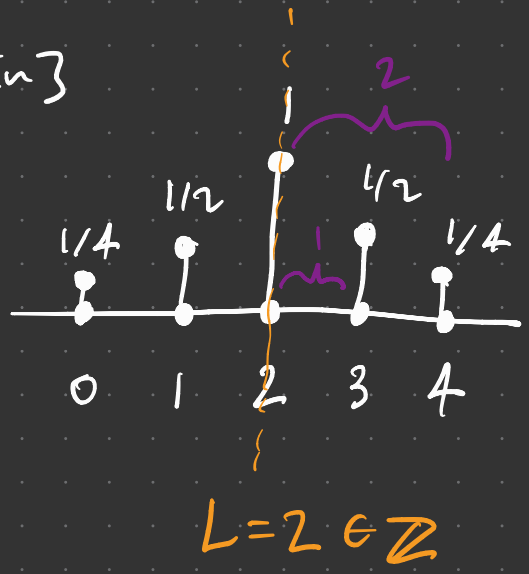
Type II : even length / symmetric

Type III : odd length / antisymmetric

Type IV : even length / antisymmetric

Type I: center of symmetry $L \in \mathbb{Z}$

Ex: $h[n]$



$$H(e^{j\omega}) = e^{-j2\omega}\left[1 + 2\left(\tfrac{1}{2}\right)\cos(\omega) + 2\left(\tfrac{1}{4}\right)\cos(2\omega)\right]$$

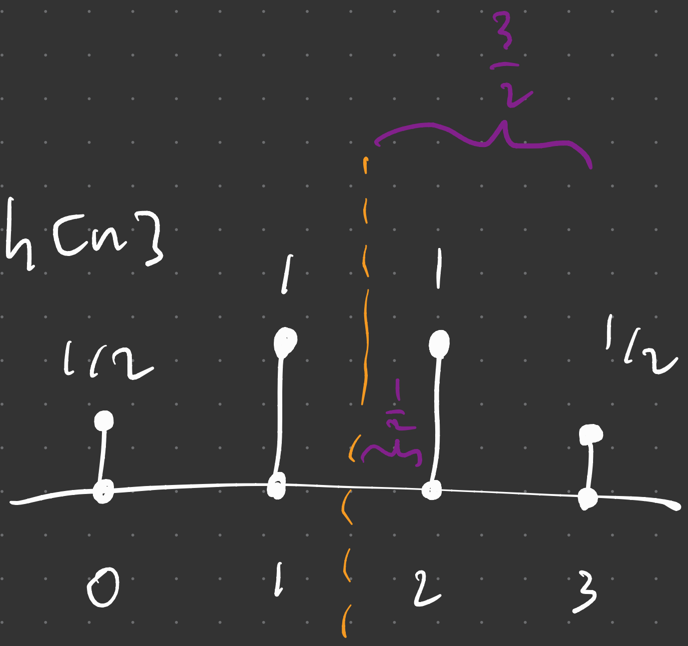
General:

$$H(e^{j\omega}) = e^{-jL\omega}\sum_{n=0}^{L} a[n]\cos(n\omega)$$

Exercise: Express $a[n]$ in terms of $h[n]$

Type II: Center of symmetry $L - \frac{1}{2} \in \mathbb{Z}$

Ex:



$L = \frac{3}{2} \Rightarrow L - \frac{1}{2} = 1 \in \mathbb{Z}$

$$H(e^{j\omega}) = e^{-j\frac{3}{2}\omega}\left[2(1)\cos\left(\frac{\omega}{2}\right) + 2\left(\frac{1}{2}\right)\cos\left(\frac{3\omega}{2}\right)\right]$$

General:

$$H(e^{j\omega}) = e^{-jL\omega}\sum_{n=0}^{L-\frac{1}{2}} b[n]\cos\left(\left(n+\frac{1}{2}\right)\omega\right)$$

Exercise: Express b[n] in terms of h[n]

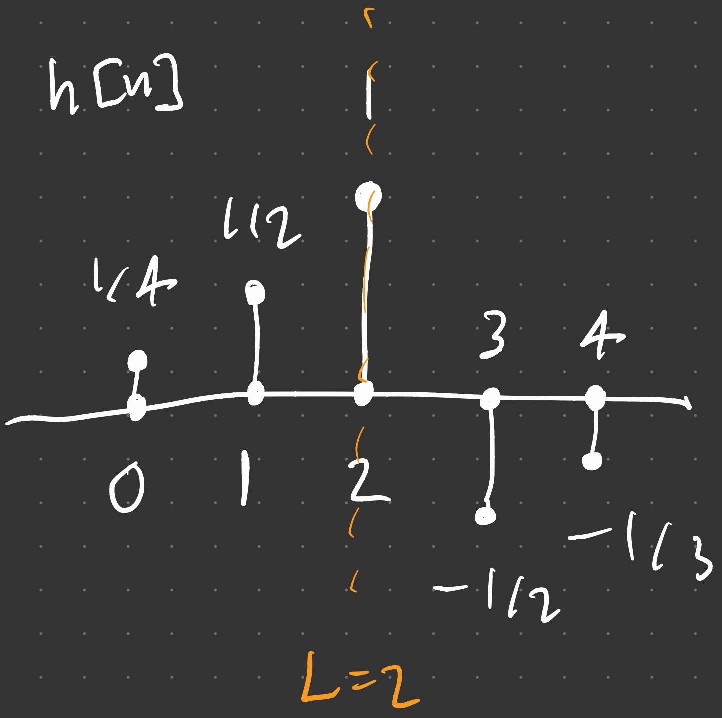
Q: what makes a filter symmetric?

A: $h[n] = h[2L - n]$

Q: what makes a filter antisymm.?

A: $h[n] = -h[2L - n]$

(Non)
Ex:



h[n] is not antisymmetric!

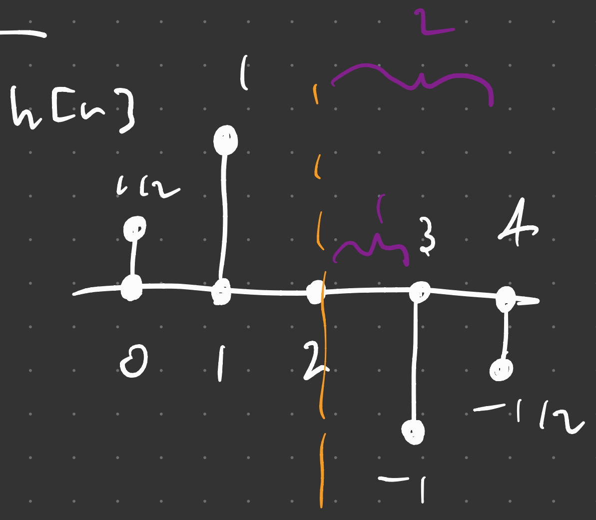
Obs: $h[n] = -h[2L - n]$

$\Rightarrow h[L] = -h[L]$

$\Rightarrow \boxed{h[L] = 0}$

Type III: Center of antisymm. $L \in \mathbb{Z}$

Ex:



$L = 2 \in \mathbb{Z}$

$$H(e^{j\omega}) = e^{-j2\omega}\left[2(1)\, j\sin(\omega) + 2\left(\tfrac{1}{2}\right) j\sin(2\omega)\right]$$

$j = e^{j\frac{\pi}{2}}$

$$= e^{j(-2\omega + \frac{\pi}{2})}\left[2\sin(\omega) + \sin(2\omega)\right]$$

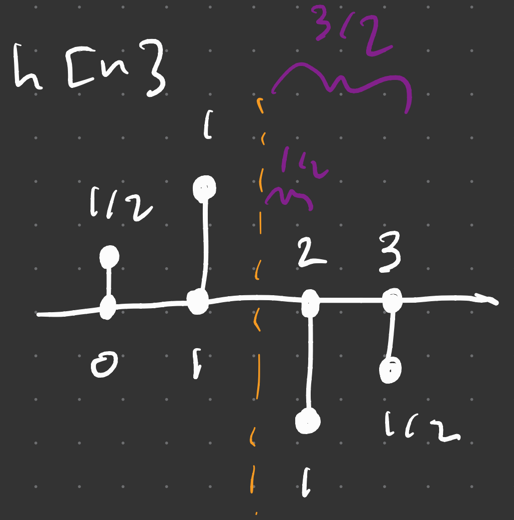
General:

$$H(e^{j\omega}) = e^{j(-L\omega + \frac{\pi}{2})} \sum_{n=0}^{L} c[n]\sin(n\omega)$$

Exercise: Express c[n] in terms of h[n]

Type IV: Center of antisymm. $L - \frac{1}{2} \in \mathbb{Z}$

Ex:



$L = \frac{3}{2} \Rightarrow L - \frac{1}{2} = 1 \in \mathbb{Z}$

$$H(e^{j\omega}) = e^{j\left(-\frac{3}{2}\omega + \frac{\pi}{2}\right)}\left[2(1)\sin\left(\frac{\omega}{2}\right) + 2\left(\frac{1}{2}\right)\sin\left(\frac{3\omega}{2}\right)\right]$$

General:

$$H(e^{j\omega}) = e^{j\left(-L\omega + \frac{\pi}{2}\right)} \sum_{n=0}^{L-\frac{1}{2}} d[n] \sin\left(\left(n+\frac{1}{2}\right)\omega\right)$$

Exercise: Express $d[n]$ in terms of $h[n]$.