Last Time: Linear-Phase Systems

Type I: center of symmetry $L \in \mathbb{Z}$
Type II: center of symmetry $L - \frac{1}{2} \in \mathbb{Z}$
Type III: center of antisymmetry $L \in \mathbb{Z}$
Type IV: center of antisymmetry $L - \frac{1}{2} \in \mathbb{Z}$

Recall: The frequency responce of a linear-phase system looks like

$$H(e^{j\omega}) = \underbrace{e^{j(-L\omega + \beta)}}_{\text{phase}} \underbrace{H_{ampl}(\omega)}_{\text{magnitude}}$$

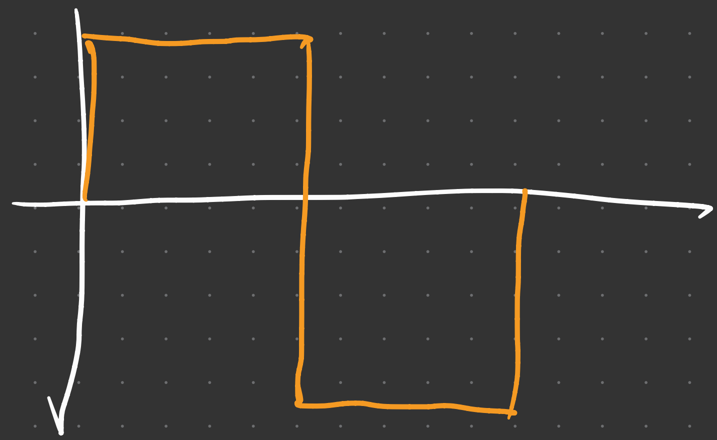
Phase Response: $\Phi(\omega) = -L\omega + \beta$ linear

Group Delay: Delay of each frequency by $h[n]$. Mathematically expressed by $-\Phi(\omega)$.
For linear-phase systems:

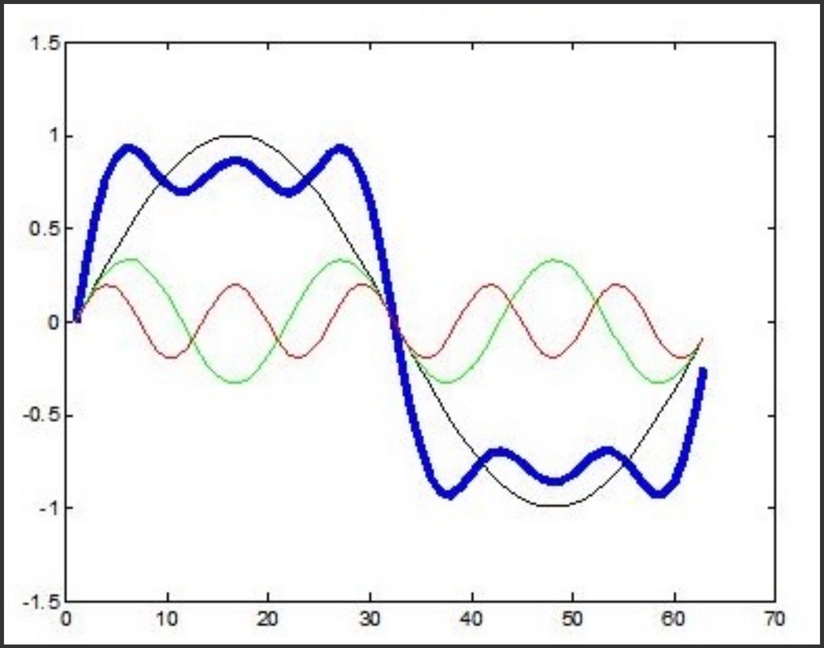
$$-\Phi'(\omega) = L \quad \text{center of } h[n]$$

Obs: Linear-phase systems have constant group delay.

Ex: Consider a square wave:

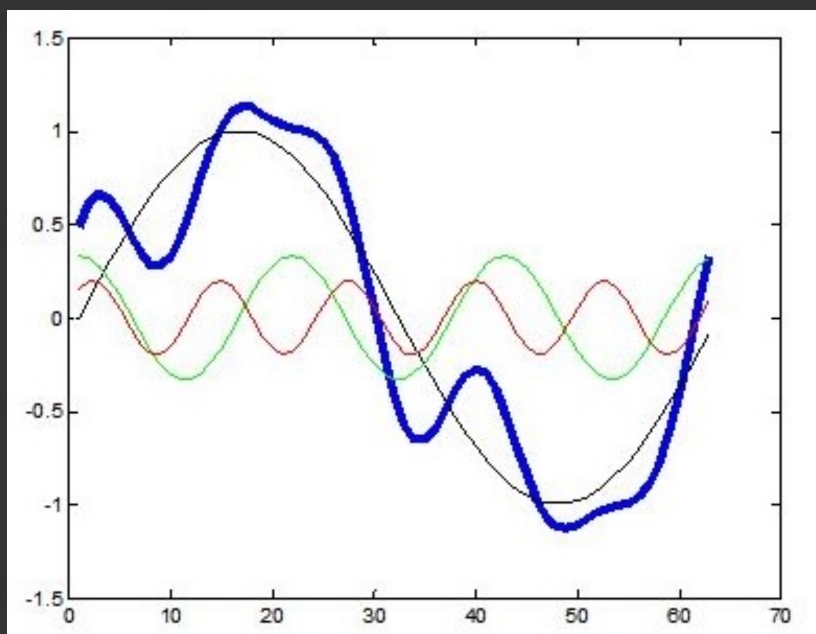


Approximate it by the first 3 nonzero terms in its Fourier series:

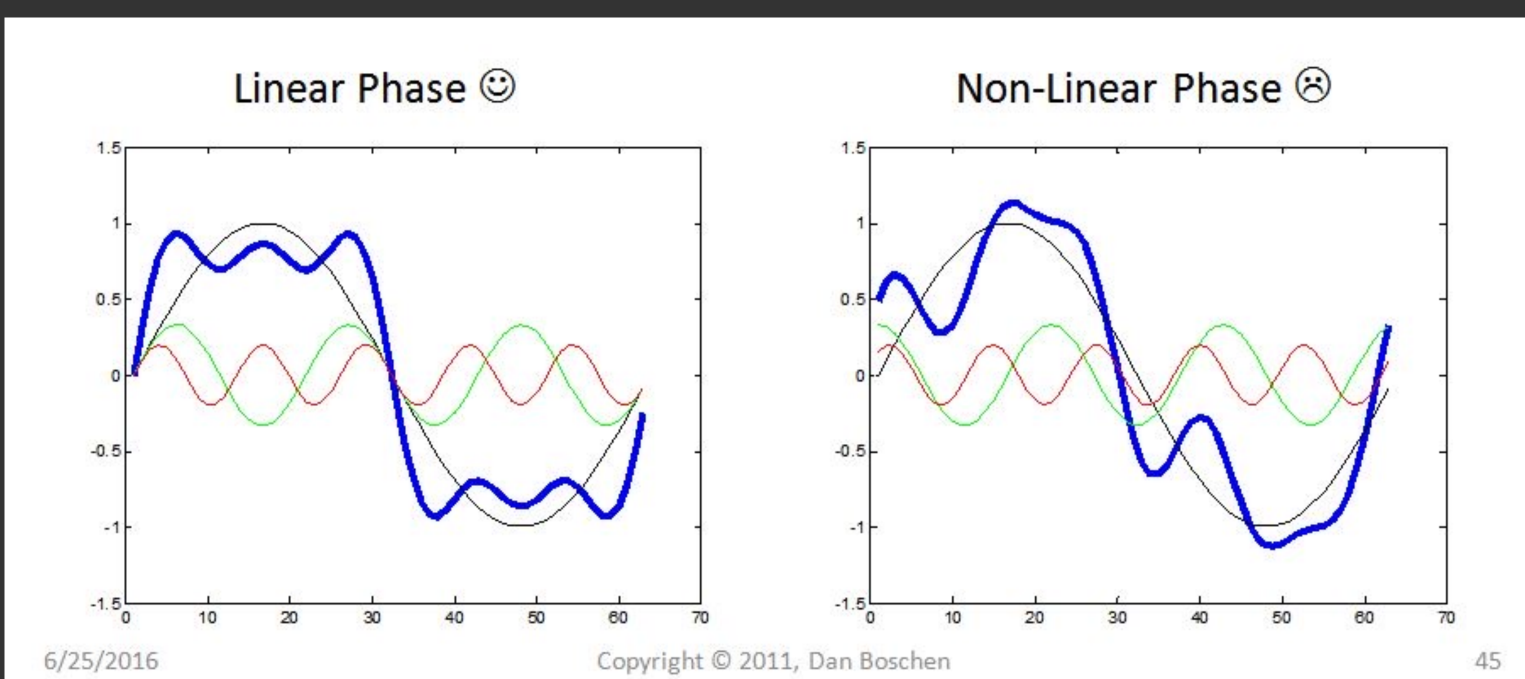


A linear-phase system will delay each pure frequency the same amount leaving the shape of the signal intact.

On the other hand, a nonlinear-phase system will shift each frequency a different amount, resulting in the <u>loss</u> of the waveform shape:

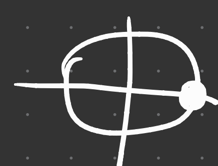


Side-by-side comparison:



Linear-Phase = Constant Group Delay

= Waveform-Preserving.

$H_{ampl}(\omega)$	$L \in \mathbb{Z}$	$L - \frac{1}{2} \in \mathbb{Z}$
Symmetric $\beta = 0$ $h[n] = h[2L-n]$	I $\sum_{n=0}^{L} a[n] \cos(n\omega)$ • No "real" constraints • "universal"	II $\sum_{n=0}^{L-\frac{1}{2}} b[n] \cos\left(\left(n+\frac{1}{2}\right)\omega\right)$  odd # of zeroes @ π • No high-pass
Anti symm. $\beta = \frac{\pi}{2}$ $h[n] = -h[2L-n]$	III $\sum_{n=0}^{L} c[n] \sin(n\omega)$  odd # of zeroes @ $0, \pi$ • only band-pass	IV $\sum_{n=0}^{L-\frac{1}{2}} d[n] \sin\left(\left(n+\frac{1}{2}\right)\omega\right)$  odd # of zeroes @ 0 • No low-pass

<u>Last Time:</u> High-pass filters cannot be Type II

$|H(e^{j\omega})|$

<u>Proof:</u>

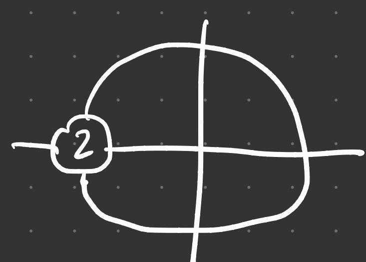
$$\cos\left(\left(n+\tfrac{1}{2}\right)\pi\right) = 0 \quad \forall n \in \mathbb{Z}$$

Obs: When designing filters, if there is any doubt, choose Type I.

Remark: Many wavelet filters are linear-phase.

Q: What happens when you cascade two Type II filters?

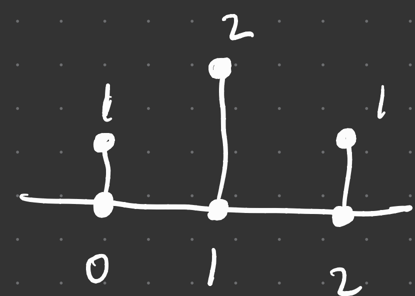
Ex: $H(e^{j\omega})$ has two zeroes @ π.



what type is this?

$$H(z) = (1+z^{-1})^2 = 1 + 2z^{-1} + z^{-2}$$

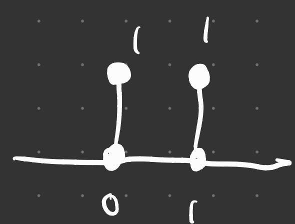
$h[n]$



$\Rightarrow$ Type I

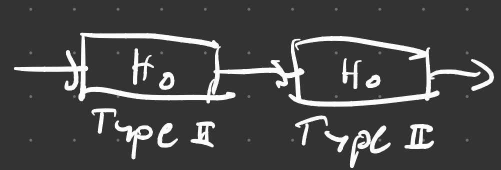
$$H(z) = H_0(z)\, H_0(z), \quad H_0(z) = 1 + z^{-1}$$

$h_0[n]$



$\Rightarrow$ Type II

Cascade



is Type I.

Claim: The cascade of two linear-phase systems of the same type is *always* Type I.

Proof: Consider the cascade

$$H(e^{j\omega}) = H_1(e^{j\omega})\, H_2(e^{j\omega})$$

with

$$H_k(e^{j\omega}) = e^{j(-L_k\omega + \beta_k)}\, H_{ampl,k}(\omega),$$

$$\boxed{\beta_k = 0 \text{ or } \frac{\pi}{2}}$$

$k = 1, 2.$

Then,

$$H(e^{j\omega}) = e^{j(-(L_1+L_2)\omega + (\beta_1+\beta_2))}\, \tilde{H}_{ampl}(\omega)$$

$$e^{j(\beta_1+\beta_2)} = \pm 1 \qquad = e^{j((L_1+L_2)\omega)}\, \tilde{\tilde{H}}_{ampl}(\omega)$$

Since the convolution of two even length filters is odd and the convolution of two odd length filters is odd, this is Type I. $\square$

Recall: For Type I and II filters

$$h[n] = h[2L-n]$$

Take z-transform of both sides:

$$H(z) = z^{-2L} H(z^{-1})$$

delay flip

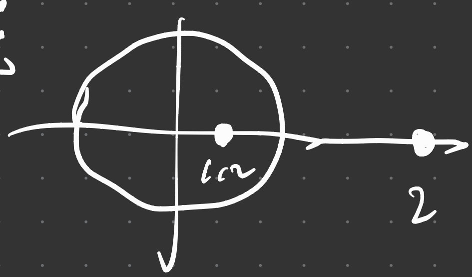
Q: How are the zeroes related?

A: Suppose z_0 is a zero.

$$0 = H(z_0) = z_0^{-2L} H(z_0^{-1})$$

Therefore z_0^{-1} is also a zero.

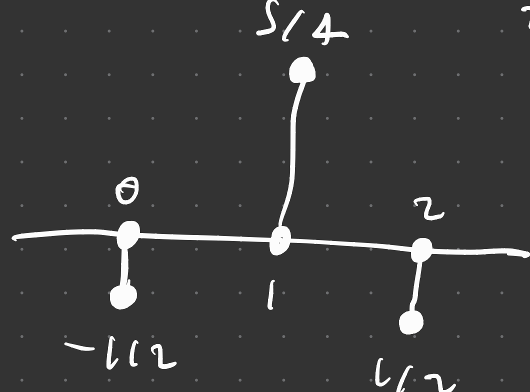
Ex:



$$H(z) = \left(1 - \frac{z^{-1}}{2}\right)\left(-\frac{1}{2} + z^{-1}\right)$$

$$= -\frac{1}{2} + \frac{5}{4} z^{-1} - \frac{1}{2} z^{-2}$$

h[n]

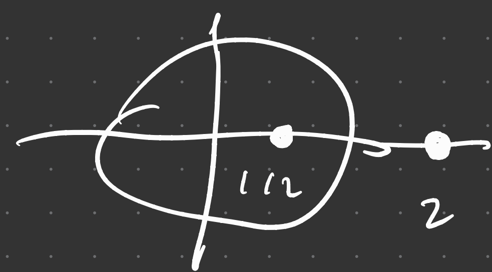


Type I

General: Four situations for the (z_0, z_0^{-1}) zero pair

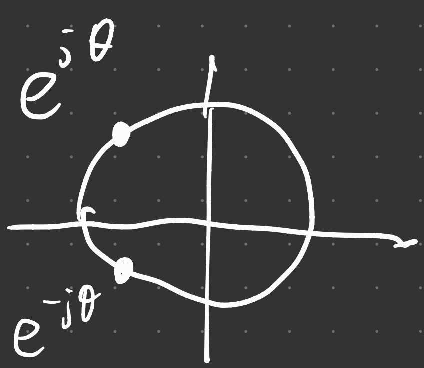
① one zero @ 1 or -1

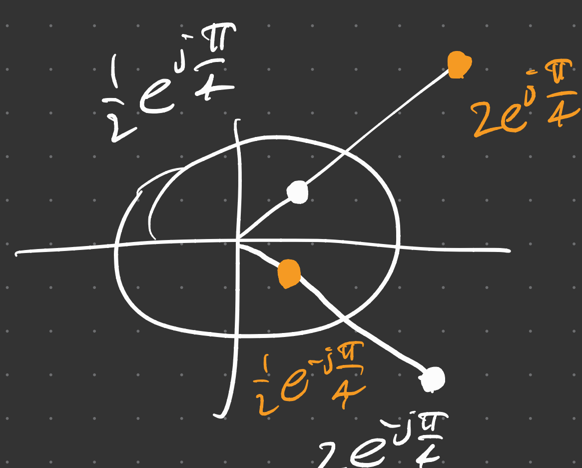
② $(r, \frac{1}{r})$, $r \in \mathbb{R}$



③ $(e^{j\theta}, e^{-j\theta})$

④ $(re^{j\theta}, \frac{1}{r}e^{-j\theta}, re^{-j\theta}, \frac{1}{r}e^{j\theta})$





Trick:

$H_1(z) = (1 - re^{j\theta} z^{-1})$ complex zero @ $z_0 = re^{j\theta}$

$H_2(z) = (-re^{j\theta} + z^{-1})$ complex zero @ $z_0^{-1} = r^{-1}e^{-j\theta}$

flip the coeffs.

add in complex conj. so that the coefficients are real.

Exercise:

Given a complex zero @ $re^{j\theta}$, determine $H(z)$ so that it is a linear-phase system with real coefficients.

All-Pass Systems

Def$^{\underline{n}}$: Given $H(e^{j\omega})$, the system is called <u>all-pass</u> if

$$|H(e^{j\omega})| = 1$$

Q: What's the point of all-pass systems?

A: To manipulate the phase response
- Minimum-phase systems
- Maximum-phase systems

Ex:
- $H(z) = 1$ identity / do nothing

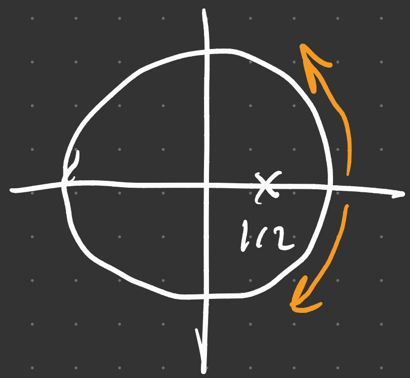
  $\Rightarrow h[n] = \delta[n]$

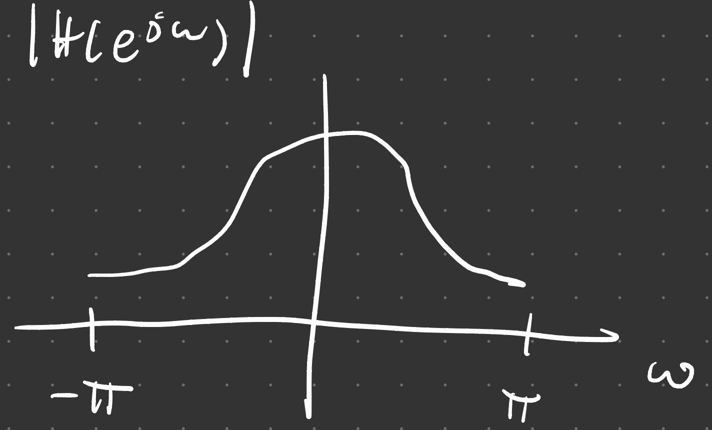
- $H(z) = z^{-N}$ delay

  $\Rightarrow h[n] = \delta[n-N]$

(Mon)

Ex:





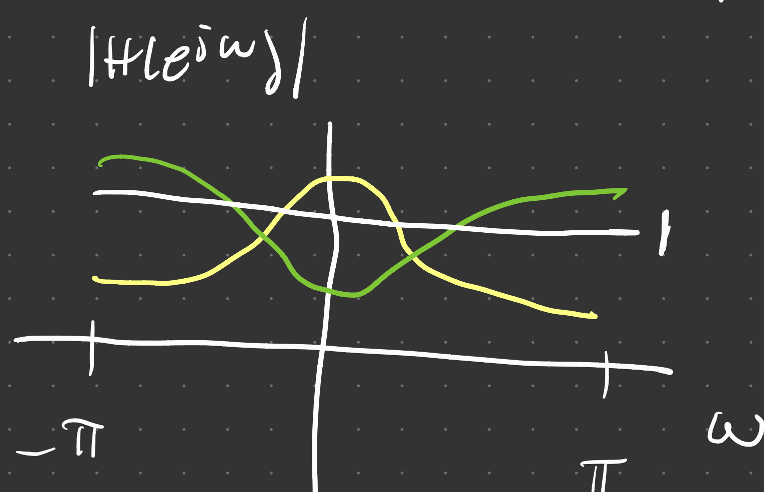
The pole "pushes" you with "strength" proportional to distance from the pole.

Obs: This system is <u>not</u> all-pass.

Q: How do we make it all-pass?

A: Add a zero to "counteract" the pole.

Ex:



All-pass systems have pole-zero pairs

Ex (cont.):

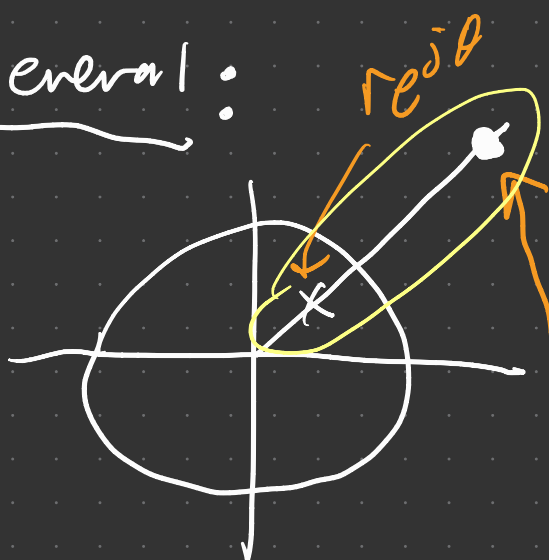
$$H(z) = \frac{-\frac{1}{2} + z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

$$H(e^{j\omega}) = \frac{-\frac{1}{2} + e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}} = \underbrace{e^{-j\omega}}_{\text{unit magnitude}} \frac{-\frac{1}{2} e^{j\omega} + 1}{1 - \frac{1}{2} e^{-j\omega}}$$

complex conj. pairs

$$\Rightarrow |H(e^{j\omega})| = 1$$

General:



$$H(z) = \frac{-re^{-j\theta} + z^{-1}}{1 - re^{j\theta} z^{-1}}$$

flip coeff and take complex conj. of denominator to get numerator.

Exercise: Where is the zero?

$$\boxed{z = r^{-1} e^{j\theta}}$$