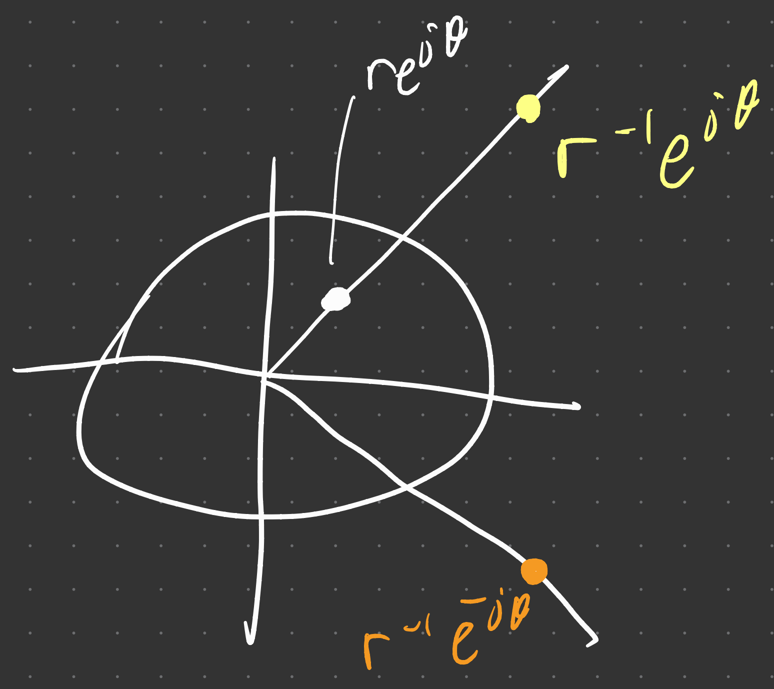
Last Time: All-Pass Systems

Review: $(1 - re^{j\theta} z^{-1}) \Rightarrow$ zero @ $re^{j\theta}$

flip coeff: $(-re^{j\theta} + z^{-1}) \Rightarrow$ zero @ $r^{-1}e^{-j\theta}$

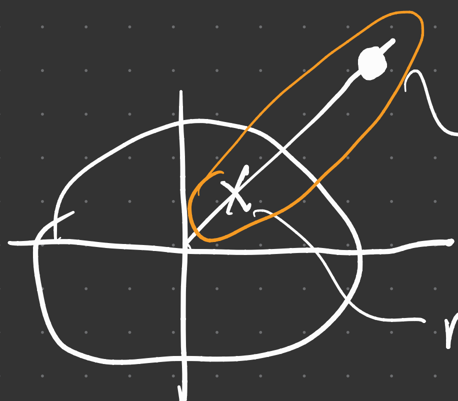
flip coef + complex conj: $(-re^{-j\theta} + z^{-1}) \Rightarrow$ zero @ $r^{-1}e^{j\theta}$



All three have the <u>same</u> magnitude resp.

Def$^{\underline{n}}$: A causal and stable system is called **first-order all-pass** if

$$H(z) = \frac{-re^{-j\theta} + z^{-1}}{1 - re^{j\theta} z^{-1}}, \quad |r| < 1$$

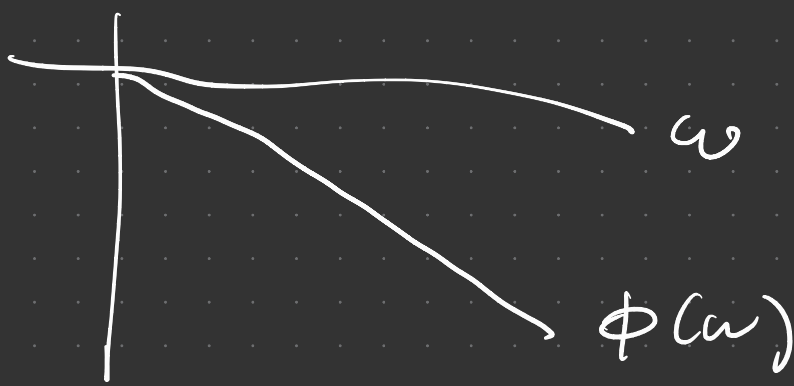


all-pass systems have pole-zero pairs

Def$^{\underline{n}}$: A causal and stable system is **Nth-order all-pass** if

$$H(z) = \prod_{k=1}^{N} \frac{-r_k e^{-j\theta_k} + z^{-1}}{1 - r_k e^{j\theta_k} z^{-1}}, \quad |r_k| < 1$$

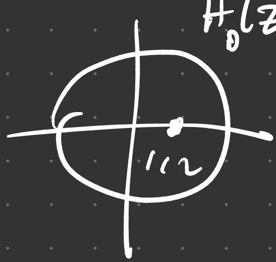
Theorem: The phase response $\Phi(\omega)$ of a stable and causal all-pass system is monotone decreasing.



Proof is long and boring ...

Exercise:

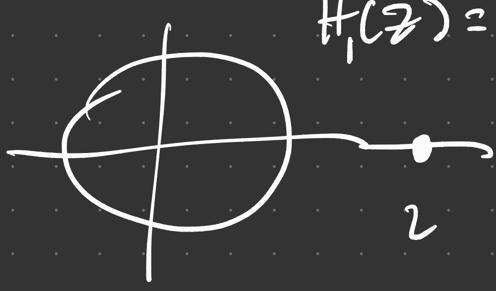
$H_0(z) = 1 - \frac{1}{2} z^{-1}$



Find all systems with same magnitude response and order.

By inspection:

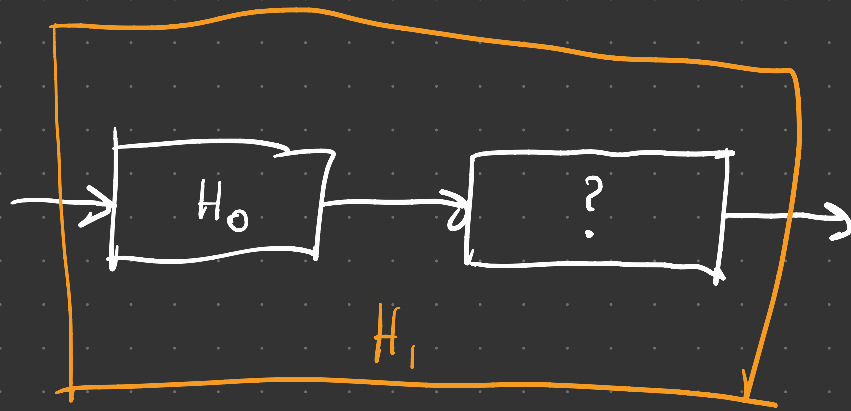
$H_1(z) = -\frac{1}{2} + z^{-1}$



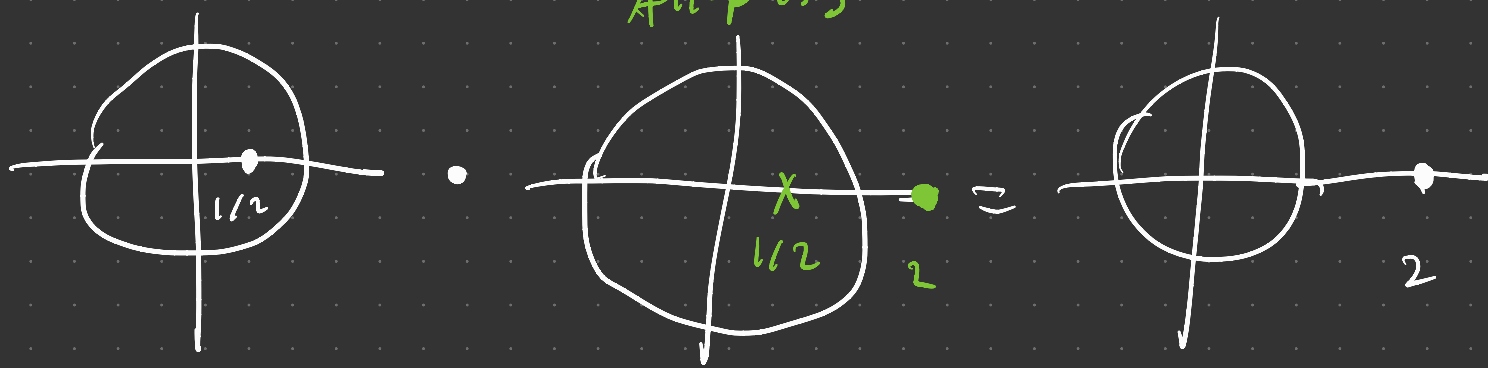
Q: What is the difference?

A: The phase.

Q: How are these two systems related?



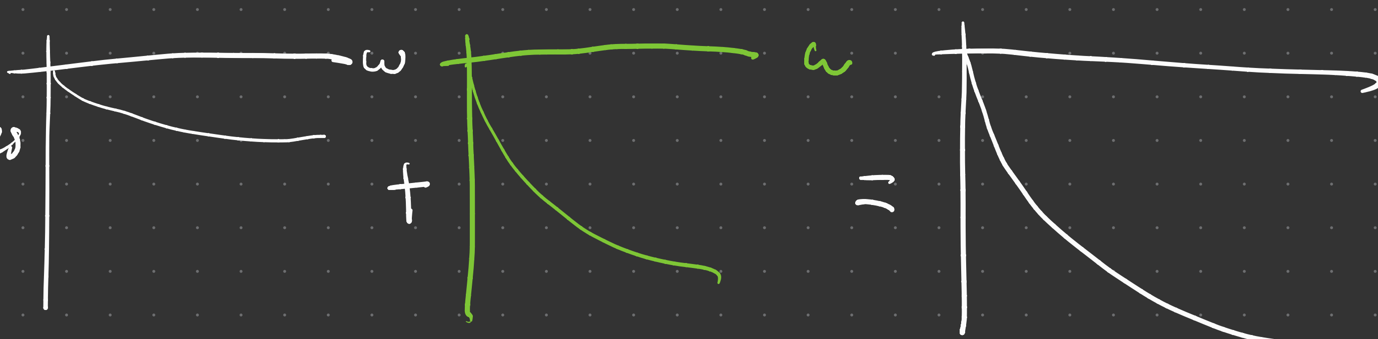
All-pass



$H_0(z)$ $\qquad$ $H(z)$ $\qquad$ $H_1(z)$

$$\left(1-\frac{1}{2}z^{-1}\right)\cdot\frac{\left(\frac{1}{2}+z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)}=\left(\frac{1}{2}+z^{-1}\right)$$

Phase responses



Minimum phase $\qquad$ Maximum phase

Obs: All-pass systems can be used to derive many new systems with the same magnitude response without changing the system effect.

- Low-pass remains low-pass
- High-pass remains high-pass
- etc.

Obs: Minimum-phase "act faster"

- smaller group delay

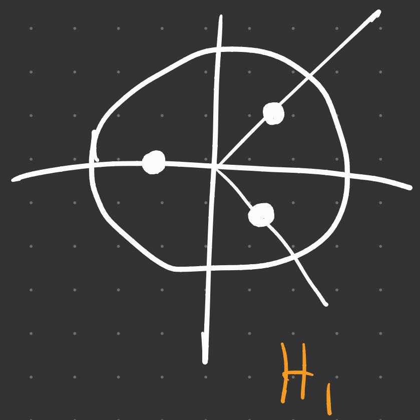
Maximum-phase "act slower"

Obs: You can make a system act faster or slower with an all-pass filter.

Remark: This is how you understand DSP

- Understand how poles/zeroes affect the system
- Understand how to manipulate them.

Exercise:



How many other systems have the same order and magnitude response?

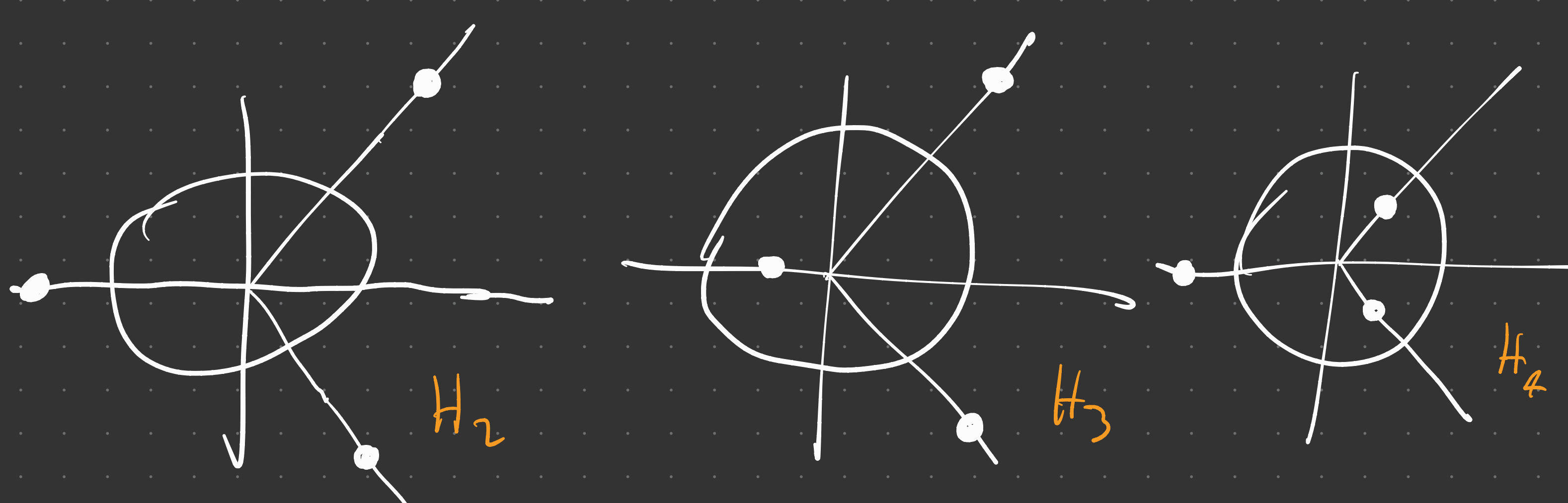
Sol$^{\underline{n}}$: Flip zeros

$\Rightarrow 2^3 = 8$ total systems.

General: 2^N, N is the # of poles/zeros

Q: How many with real coefficients?

A: 4, The two complex zeros need to flip together.



Q: Which one is

- Minimum-phase ?
- Maximum-phase ?

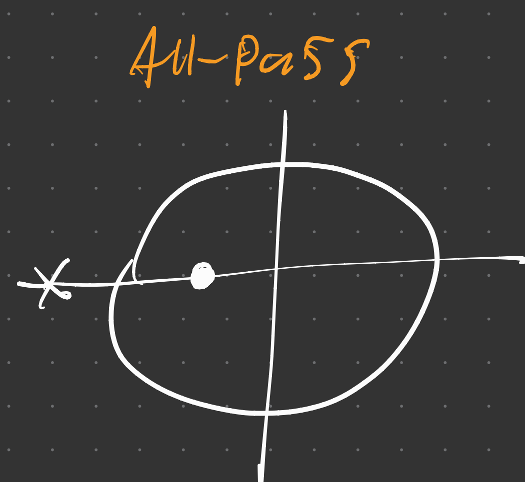
A: Minimum-phase $= H_1$

Maximum-phase $= H_2$

Why? By flipping a zero inside the unit circle, you decrease the phase

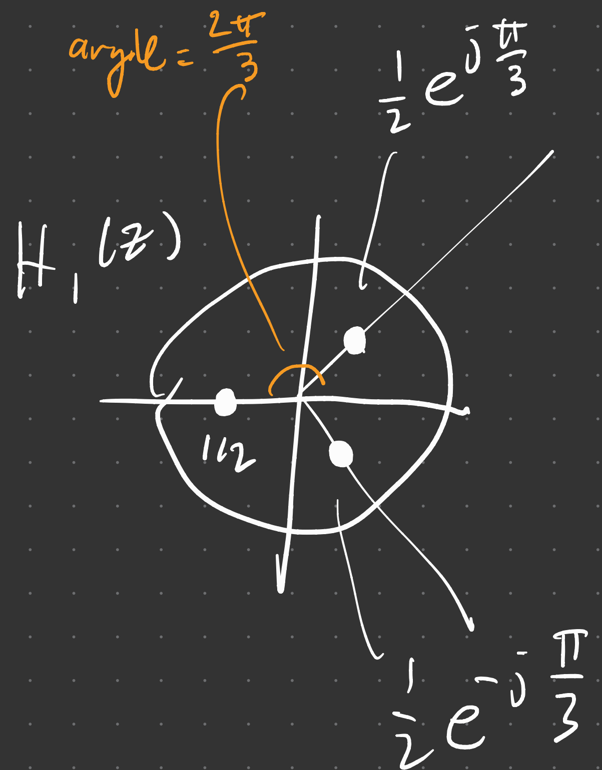
Proof: Previous exercise.

Obs:

$H_4 \cdot$ 

$= H_1$

Obs:

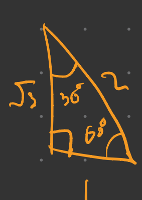


$$H_1(z) = \left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}e^{j\frac{\pi}{3}}z^{-1}\right)\left(1-\frac{1}{2}e^{-j\frac{\pi}{3}}z^{-1}\right)$$

$$= \left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}e^{-j\frac{\pi}{3}}z^{-1}-\frac{1}{2}e^{j\frac{\pi}{3}}z^{-1}+\frac{1}{4}e^{0}z^{-2}\right)$$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

$$= \left(1+\frac{1}{2}z^{-1}\right)\left(1-\cos\left(\frac{\pi}{3}\right)z^{-1}+\frac{1}{4}z^{-2}\right)$$



$$= \left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)$$

	1	$-\frac{1}{2}$	$\frac{1}{4}$
		1	$\frac{1}{2}$
	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{8}$
1	$-\frac{1}{2}$	$\frac{1}{4}$	
1	0	0	$\frac{1}{8}$

$$= 1+\frac{1}{8}z^{-3}$$

Q: Why so many zero coeffs?

A: Symmetry.

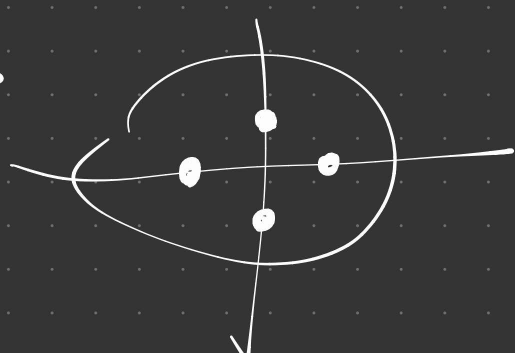
$$H_1(z) = H_1\left(ze^{-j\frac{2\pi}{3}}\right) = H_1\left(ze^{-j\frac{4\pi}{3}}\right)$$

$$a + bz^{-1} + cz^{-2} + dz^{-3} = H_1(z)$$

$$a + bz^{-1}e^{j\frac{2\pi}{3}} + cz^{-2}e^{j\frac{4\pi}{3}} + dz^{-3}\underbrace{\cancel{e^{j\frac{6\pi}{3}}}}_{1} = H_1\left(ze^{-j\frac{2\pi}{3}}\right)$$

$$\Rightarrow b = c = 0$$

Ex:



$$H(z) = a + bz^{-4}$$

Alternative Characterization of Min/Max-Phase

Define the <u>partial energy</u>

$$E[n] = \sum_{k=-\infty}^{n} |h[k]|^2$$

<u>Theorem:</u> For all systems of same order and magnitude

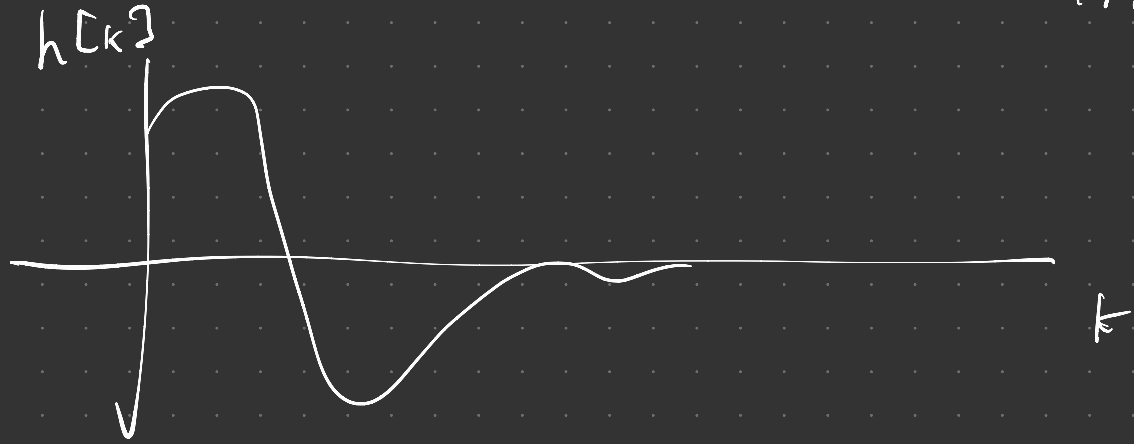
$$\sum_{k=-\infty}^{n} |h[n]|^2 \leq \sum_{k=-\infty}^{n} |h_{min}[n]|^2, \quad \forall n \in \mathbb{Z}$$

where $h_{min}[n]$ is the min-phase and $h[n]$ is any other system in this family, i.e., h_{min} has the <u>largest</u> partial energy.

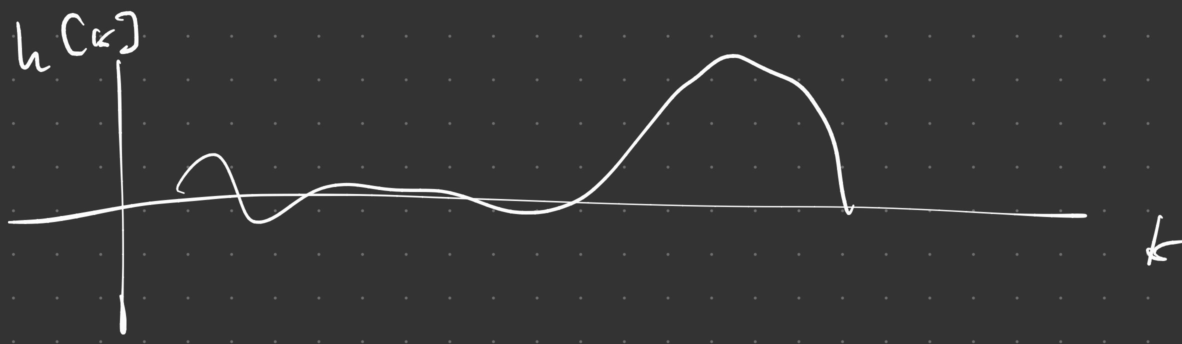
Theorem: The max-phase system $h_{max}[n]$ has the smallest partial energy.

$$\sum_{k=-\infty}^{n} |h_{max}[k]|^2 \leq \sum_{k=-\infty}^{n} h[k] \quad \forall n \in \mathbb{Z}$$

Obs: • Min-phase has energy concentrated in the "front"



• Max-phase has energy concentrated in the "back"



Obs: Min-phase systems act faster.

Remark: You can make a system act faster or slower by manipulating the phase.

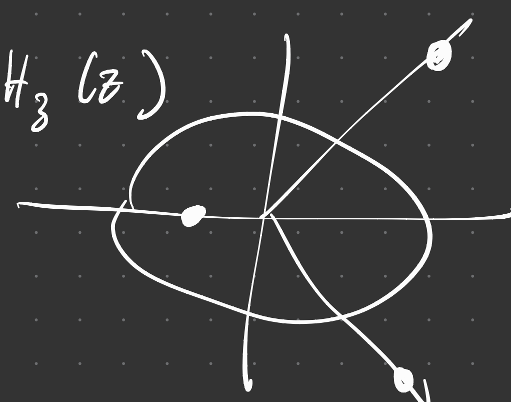
Exer: Compare partial energy of

$h_1[n]$ and $h_3[n]$

min-phase

$H_1(z)$

$H_3(z)$



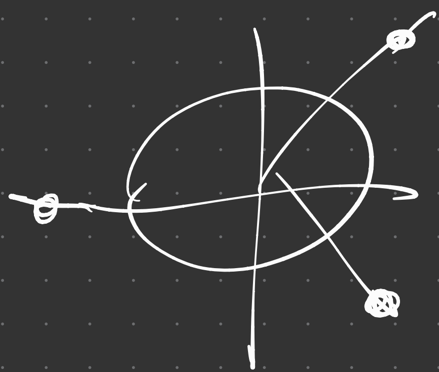
$H_1(z) = 1 + \frac{1}{8} z^{-3}$ $\quad$ $H_3(z) = \frac{1}{4} - \frac{3}{8} z^{-1} + \frac{3}{4} z^{-2} + \frac{1}{2} z^{-3}$

n	$E_1[n]$
0	1
1	1
2	1
3	$1 + \frac{1}{64}$

Parseval's Theorem

n	$E_3[n]$	
0	$\frac{1}{16}$	≤ 1
1	$\frac{1}{16} + \frac{9}{64}$	≤ 1
2	$\frac{1}{16} + \frac{9}{64} + \frac{9}{16}$	≤ 1
3	$1 + \frac{1}{64}$	$\leq 1 + \frac{1}{64}$

Exer: Compae with $h_2[n]$ max-phase

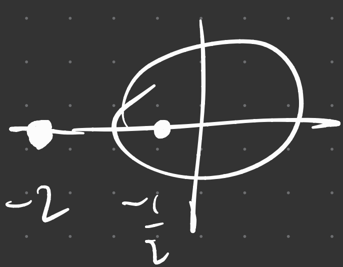


$$H_3(z) = \frac{1}{8} + z^{-3}$$

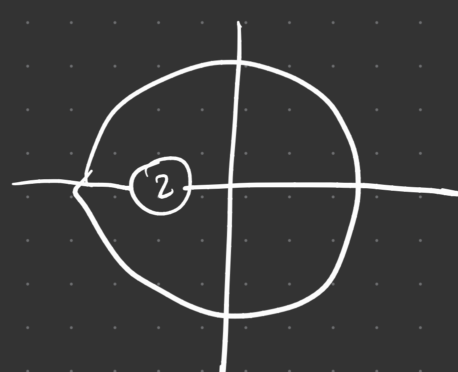
n	$E_3[n]$
0	$\frac{1}{64}$
1	$\frac{1}{64}$
2	$\frac{1}{64}$
3	$1+\frac{1}{64}$

Exer: Compare with $h_4[n]$

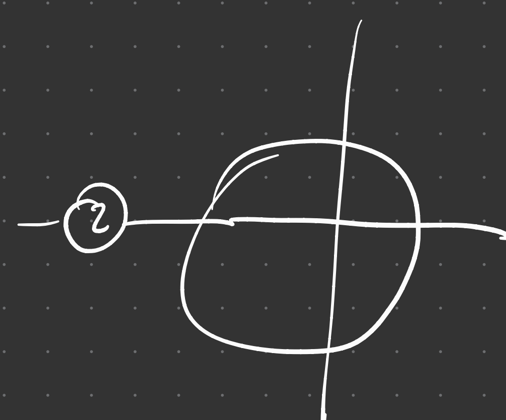
Ex:



linear phase



min-phase



max-phase

Multirate Systems

Ex: • YouTube full screen

increased the rate

• Watching videos at 2x speed.

Recall:

$x[n] = x_c(nT)$, $\frac{1}{T}$ sampling rate

Down sample by factor 2:

Slower

$(\downarrow 2)\, x[n] = x[2n] = x_c(2nT)$, $\frac{1}{2T}$ sampling rate

Intuition:

Downsampling <u>reduces</u> bandwidth

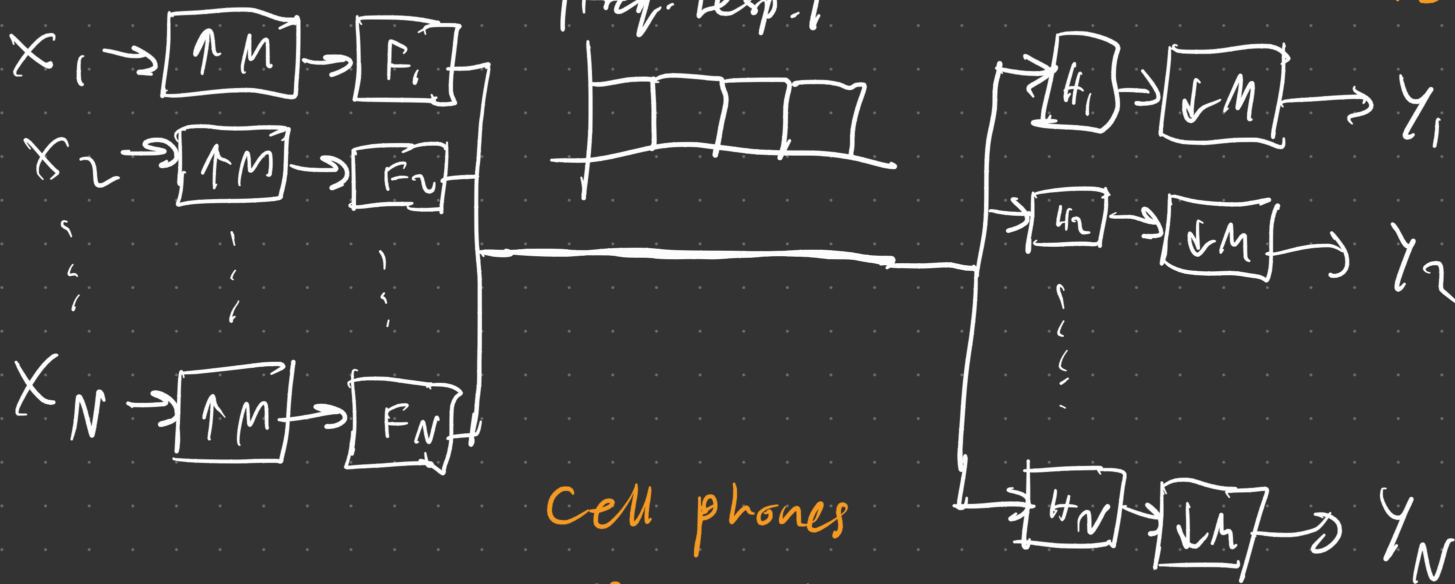
Upsampling <u>inceases</u> bandwidth

Ex:

Filter bank

Voice is sampled at 8 kHz

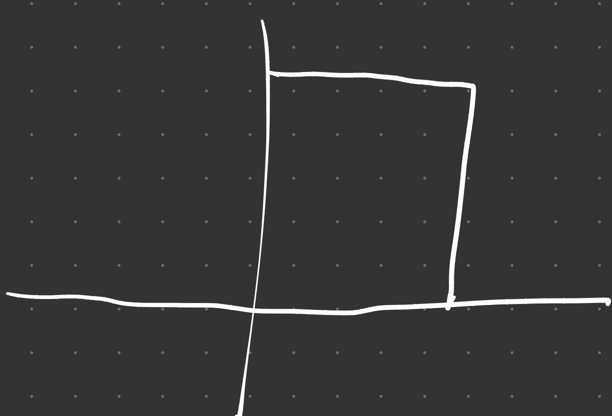
8 kHz



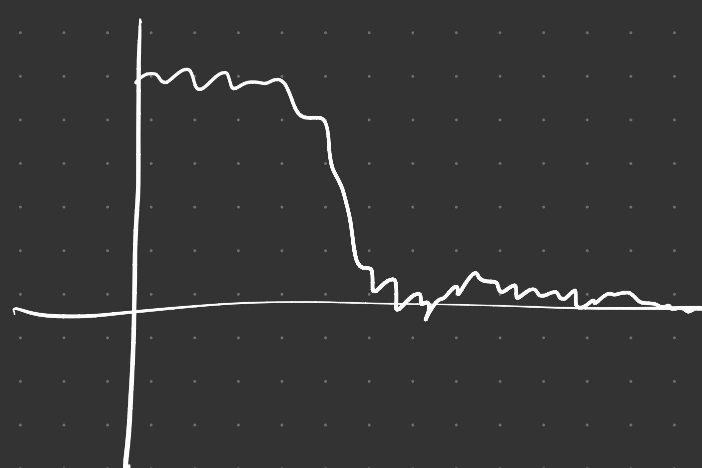
Cell phones operate in MHz or GHz (RF spectrum)

Also how cable TV works

$|H_1(e^{j\omega})|$



ideal filter



actual filter

Upsampling & Interp by factor 2

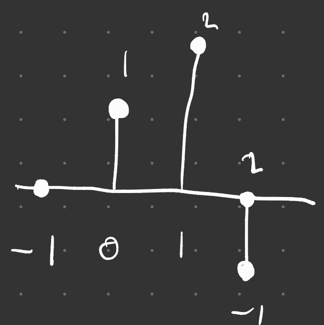
Q: What is upsampling?

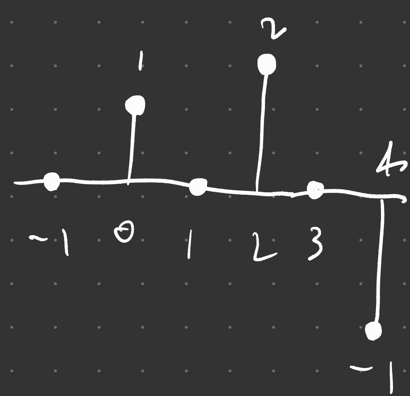
A: Putting zeros between samples

Ex:

upsampling is always followed by interpolation

$$v[n] \to \boxed{\uparrow 2} \xrightarrow{u[n]} \boxed{H} \to y[n]$$





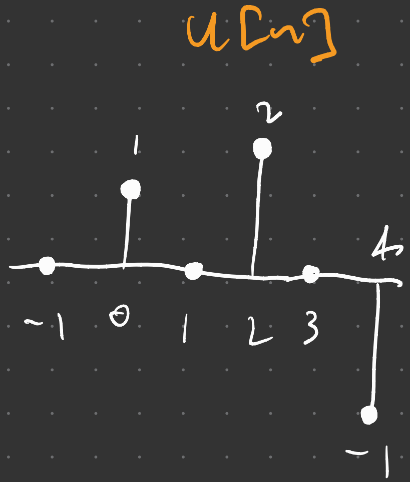
not useful e.g. in an image this would be dark lines

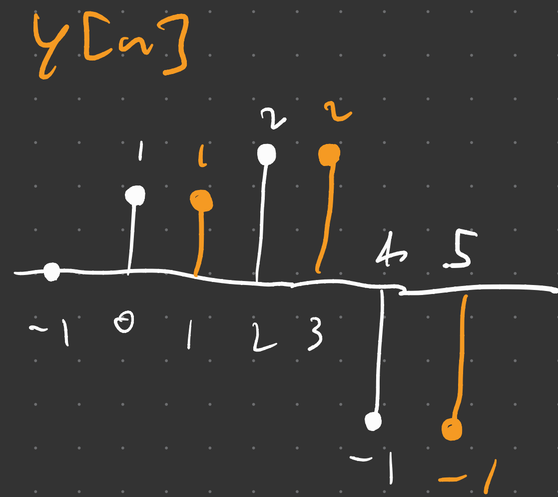
Q: what is the job of the filter?

A: To interpolate the zeros.

Q: what's the easiest way to interpolate?

A: Repeat value to the left.



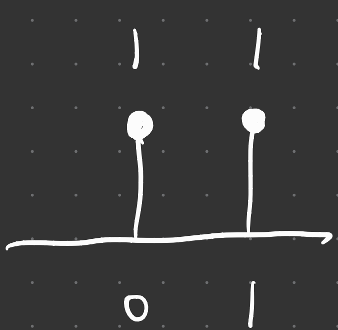


Exercise: What is $h[n]$ such that

$$(h * u)[n] = y[n] \text{ ?}$$

Soln:

$h[n]$



Why?:

$$y[n] = u[n] + u[n-1]$$

$$h[n] = \delta[n] + \delta[n-1]$$