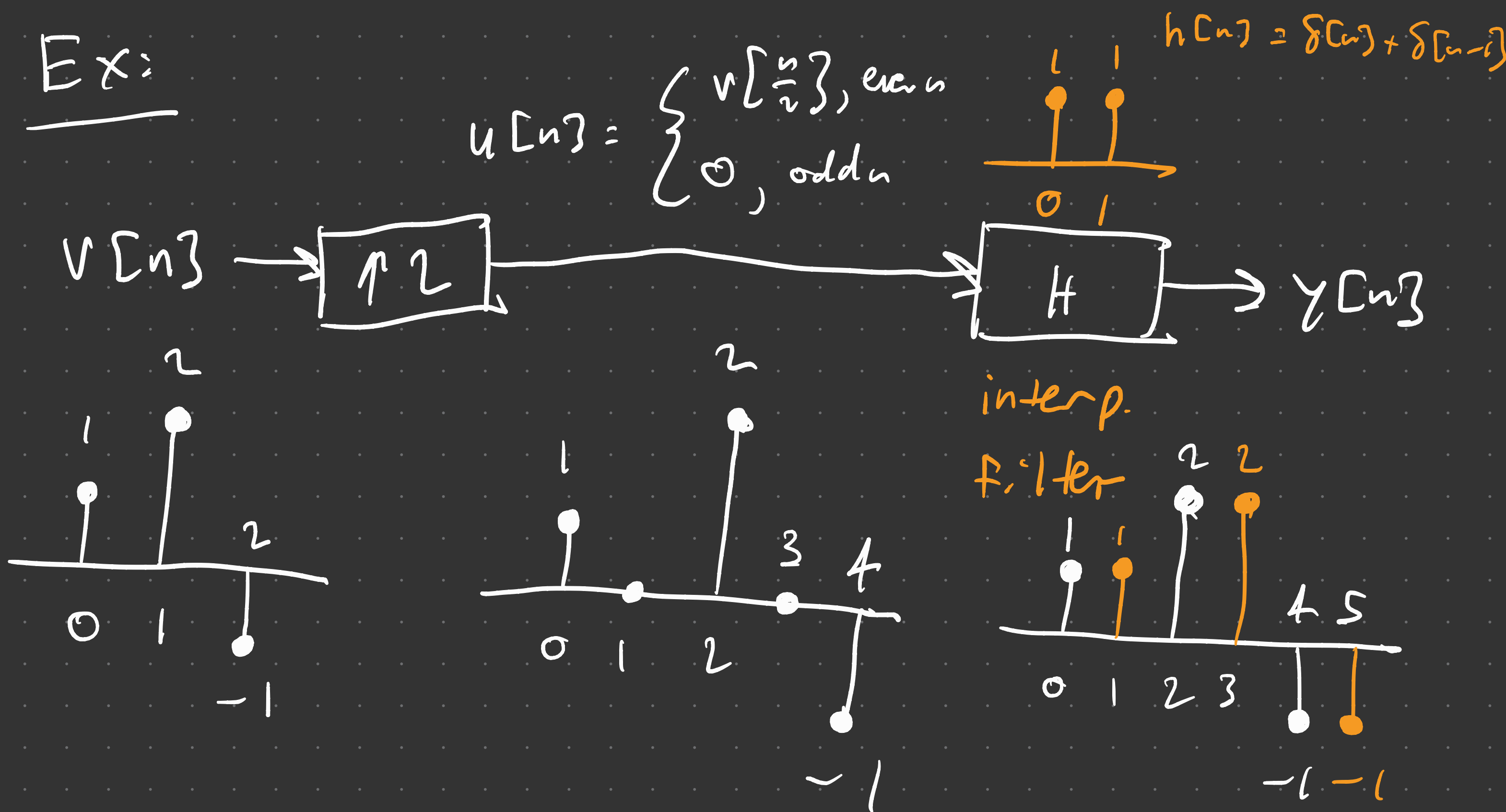
Last Time: Multirate Systems

Ex:



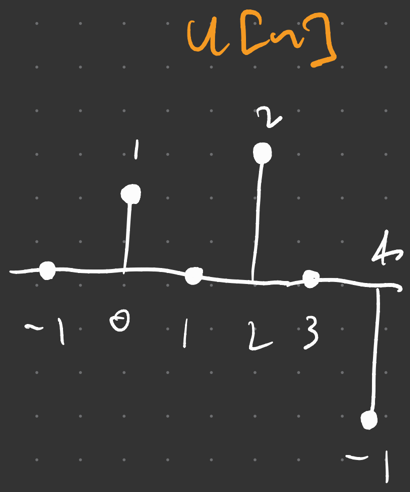
Q: Why is this bad?

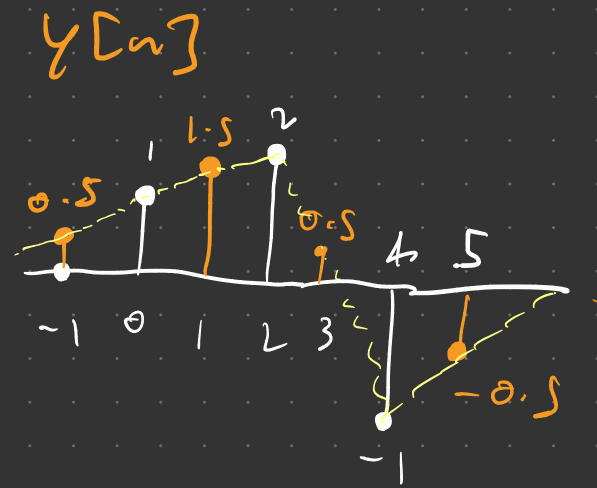
A: Blocking artifacts.

(Imagine upsampling by factor 8)

Remark: Interpolation filter design is a kind of art.

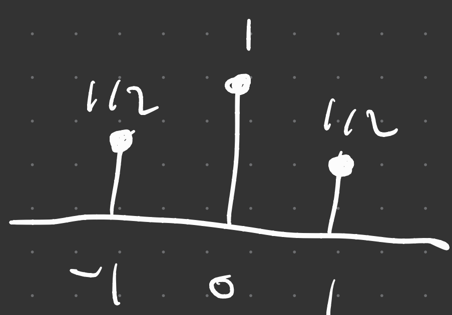
Q: What about linear interpolation?





Exercise: What is $h[n]$?

Sol$^{\underline{n}}$:



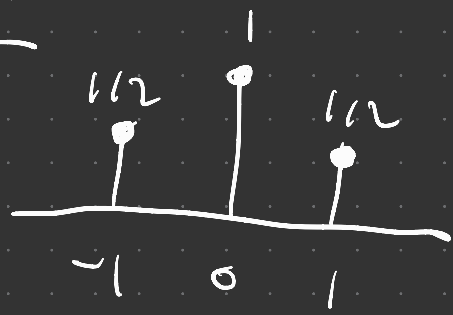
$$y[n] = \frac{u[n+1] + u[n-1]}{2} + u[n]$$

$$h[n] = \frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]$$

Q: Why is this bad?

A: Looks too sharp. We want something that looks smooth.

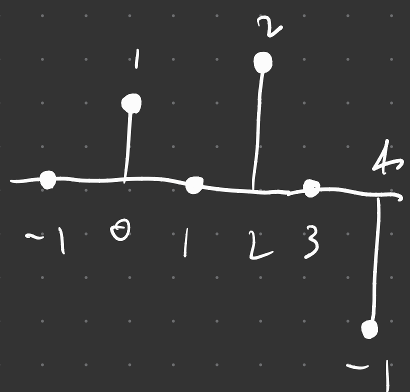
Obs:



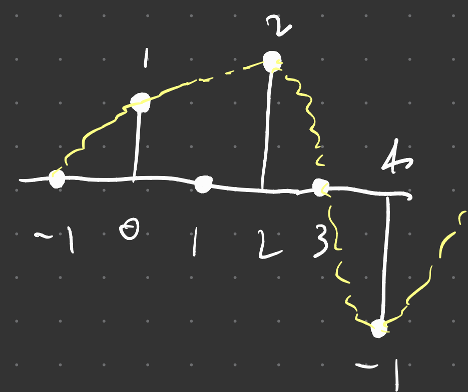
looks kinda like a sinc $\Rightarrow$ low-pass

Goal:

$u[n]$

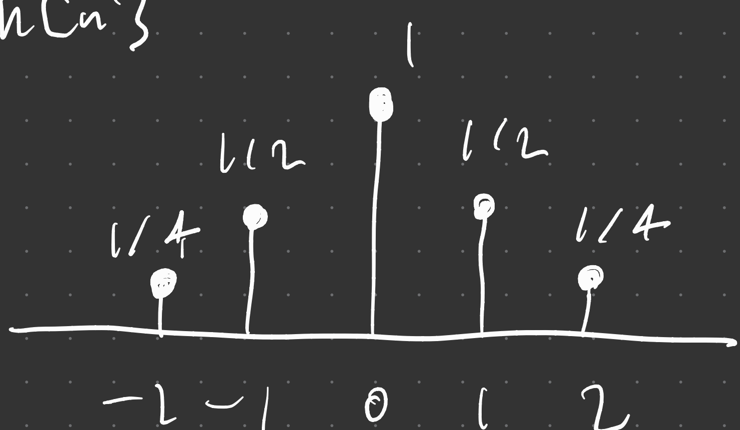


$y[n]$



Exercise: What about the filter

$h[n]$



?

Obs: It's bad because it changes the original samples!

Check what happens @ $y[0]$

$$y[0] = 1 + 0 + \frac{1}{4}(2) = \frac{3}{2} \neq u[0]$$

we destroyed the original samples!

Goal: Interpolate and preserve samples.

Obs: The issue is at even coeff. of $h[n]$.

Property: An interpolation filter must satisfy

$$h[2n] = \delta[n]$$

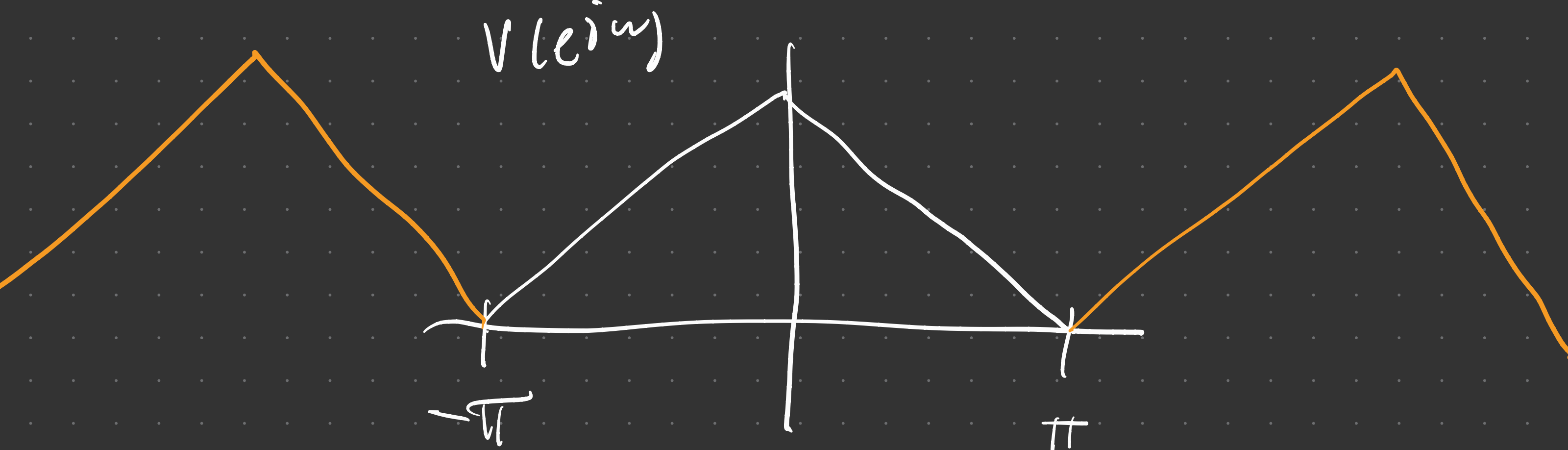
$h[2n] \rightarrow (\downarrow 2)\, h[n]$

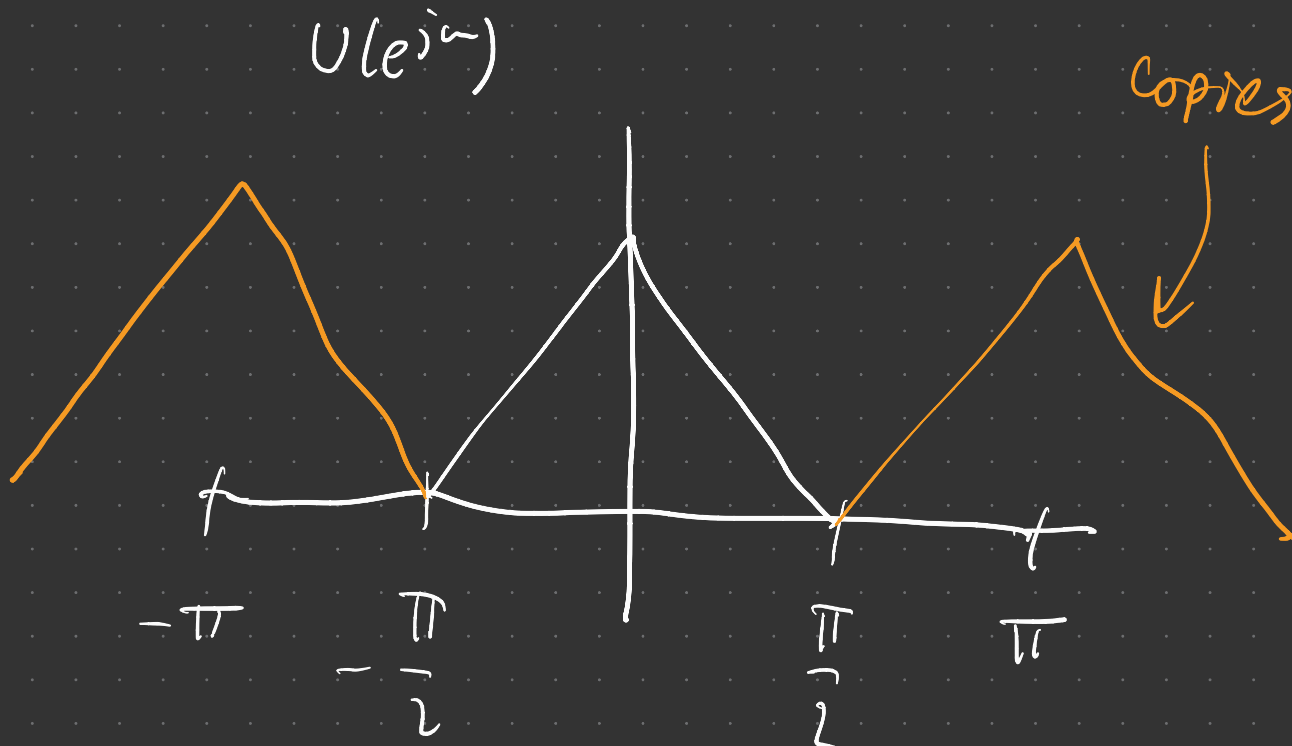
half-band condition

Obs: Interpolation filter design is only about the odd coeff.

Frequency-Domain Analysis

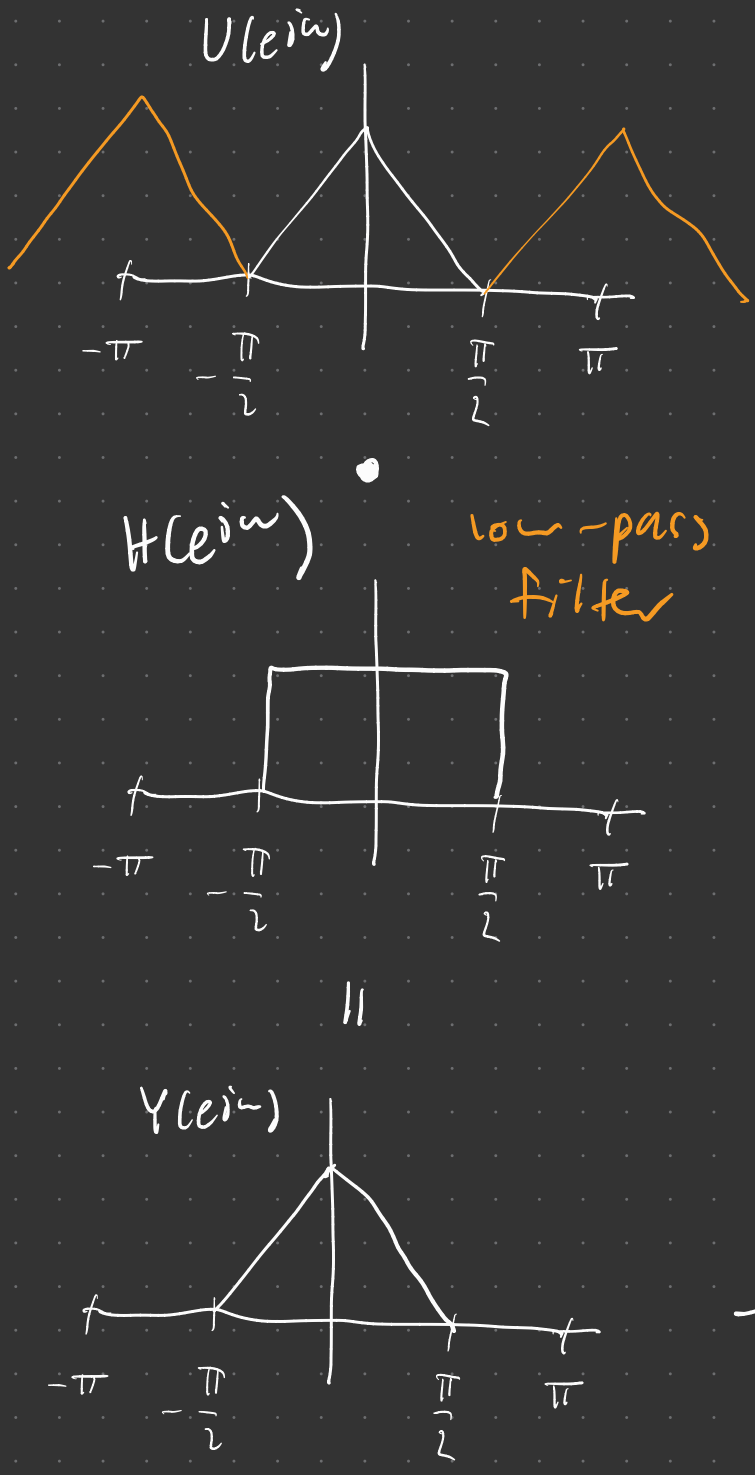
$$U(z) = V(z^2) \Rightarrow U(e^{j\omega}) = V(e^{j2\omega})$$



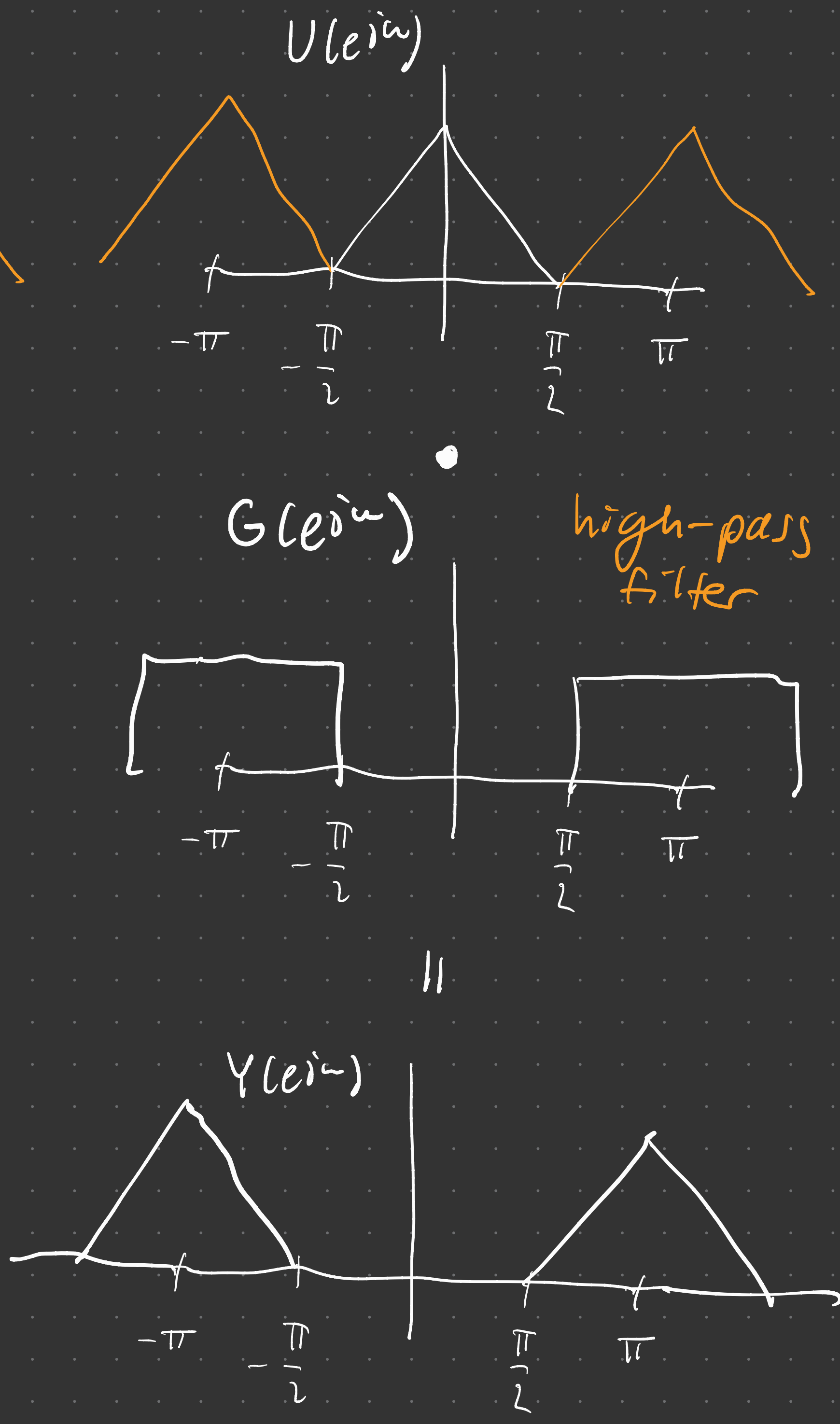


Remark: We don't want the spectrum to look like this!

Two Options:



Low-pass Interp.



High-pass Interp.

We have already seen low-pass interpolation.

Q: What is high-pass interpolation?

Q: Given a low-pass filter, is there a corresponding high-pass filter?

A: Yes.

$$G(e^{j\omega}) = H(e^{j(\omega-\pi)})$$ shift in freq.

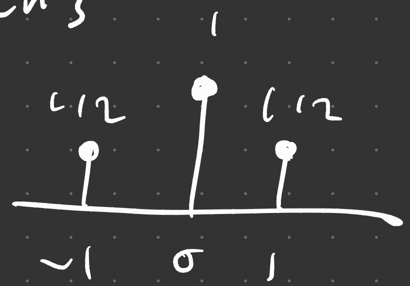
$$g[n] = e^{j\pi n} h[n]$$ modulation in time

$$= (-1)^n h[n]$$

Obs: If $h[n]$ satisfies half band condition, then so does $g[n]$. (even index stays the same).

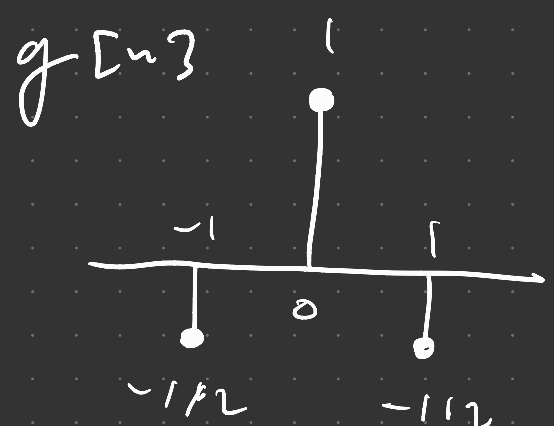
Trick: Flip every other coeff of $h[n]$ to get $g[n]$.

Exercise: Given the linear interp. filter

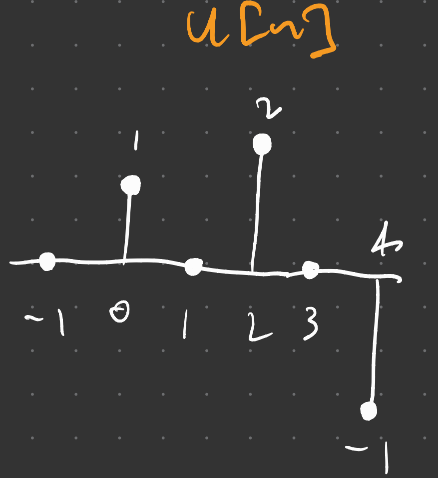


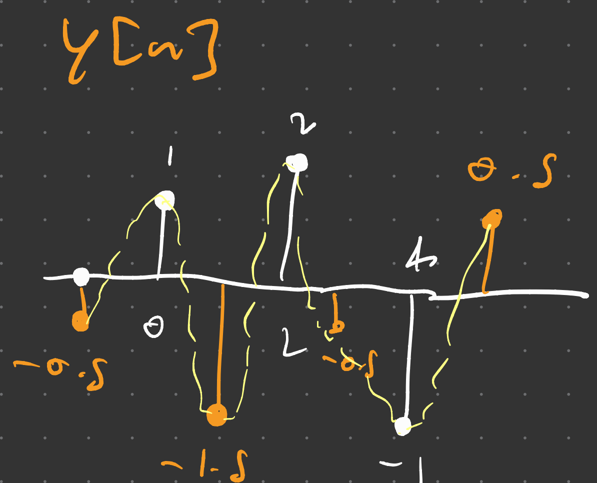
What is the corresponding high-pass filter g[n]?

Soln:



really wiggly
⇒ high-pass





Obs: Original samples are preserved because g[n] satisfies the half-band condition.

Q: Is upsampling linear?

A: Yes

$\alpha, \beta \in \mathbb{R}$

Proof:

$$(\uparrow 2)(\alpha v_1[n] + \beta v_2[n])$$

$$= \begin{cases} \alpha v_1[\frac{n}{2}] + \beta v_2[\frac{n}{2}], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$= \alpha \begin{cases} v_1[\frac{n}{2}], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} + \beta \begin{cases} v_2[\frac{n}{2}], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

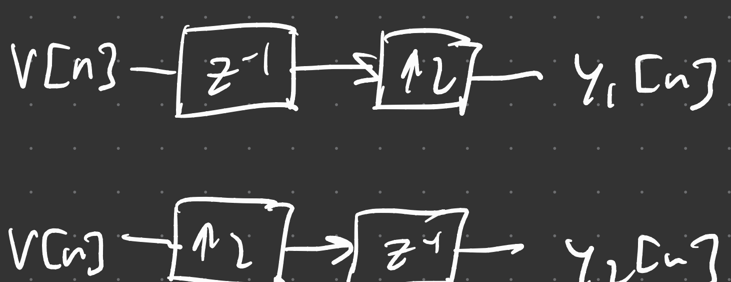
$$= \alpha (\uparrow 2) v_1[n] + \beta (\uparrow 2) v_2[n]$$

□

Q: Is upsampling time-invariant?

A: No.

Proof: After upsampling, the signal always has zero for odd coeff.



$y_1[n]$: odd coeff zero

$y_2[n]$: even coeff zero

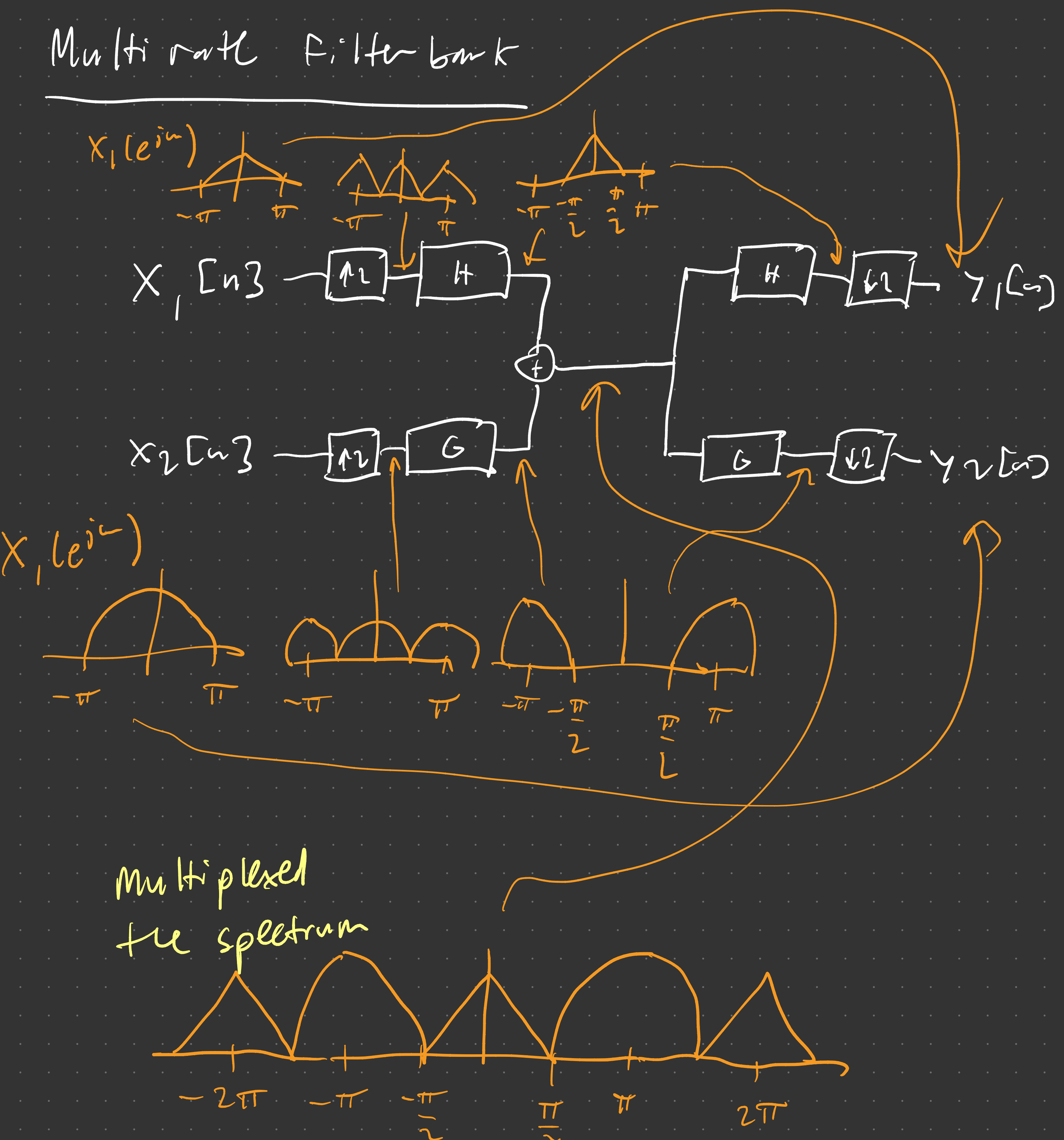
$y_1[n] \neq y_2[n]$

$\Rightarrow$ not time-invariant

□

Remark: This is why multirate DSP is hard: Operations don't commute.

Multirate filter bank



General Case of Upsampling

$\frac{n}{L} \in \mathbb{Z}$

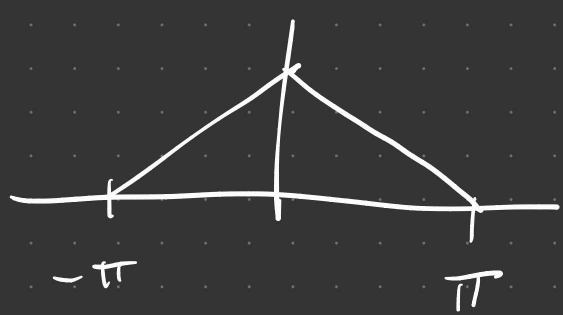
$$u[n] = \begin{cases} v\left[\frac{n}{L}\right], & n \text{ is mult. of } L \\ 0, & \text{else} \end{cases}$$

insert (L−1) zeros between samples

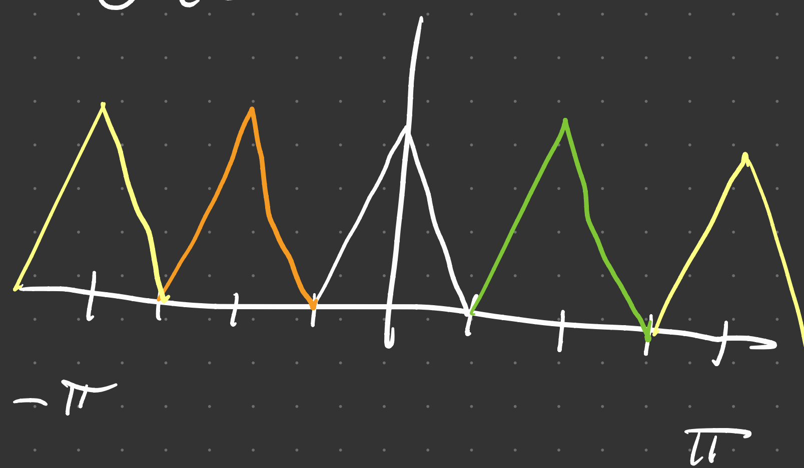
$$U(z) = V(z^L) \Rightarrow U(e^{j\omega}) = V(e^{jL\omega})$$

Ex: $L = 4$

$V(e^{j\omega})$



$U(e^{j\omega})$



- 3 copies of the spectrum
- 4 total spectrums

Obs: 4 options for the interp. filter

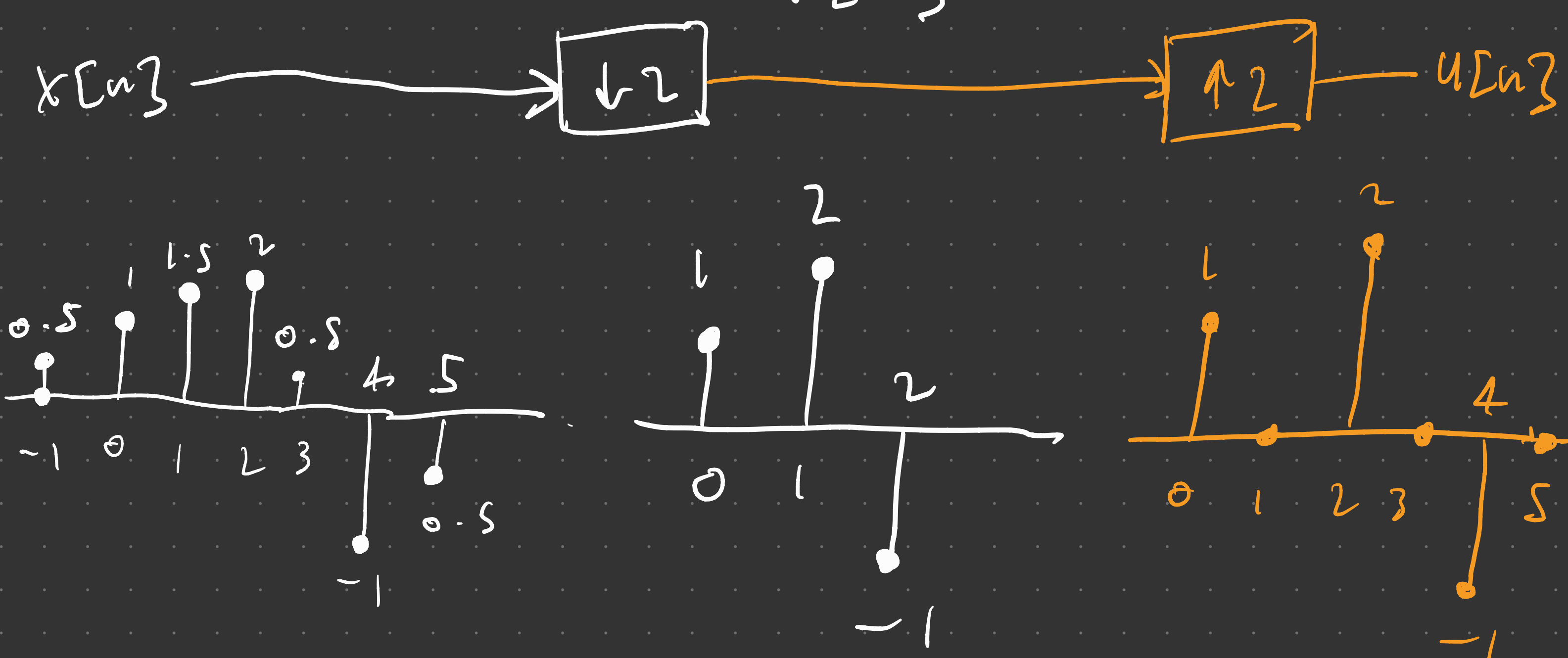
General: L interp. filters for the L spectrums

Down samplings by Factor 2

Q: What is down sampling?

A: Deleting samples

$v[n] = x[2n]$



Q: Is down sampling linear?

A: Yes

Q: Is it time-invariant?

A: No

Q: Is it invertible?

A: No.

Q: How are the spectrums $X(e^{j\omega})$ and $V(e^{j\omega})$ related?

A: $U(e^{j\omega}) = V(e^{j2\omega})$

How are $u[n]$ and $x[n]$ related?

$$
\begin{array}{lcccccc}
 & x[n]: & x[0] & x[1] & x[2] & x[3] & x[4] \dots \\
+ & (-1)^n x[n]: & x[0] & -x[1] & x[2] & -x[3] & x[4] \\
\hline
 & 2u[n]: & 2x[0] & 0 & 2x[2] & 0 & 2x[4] \dots
\end{array}
$$

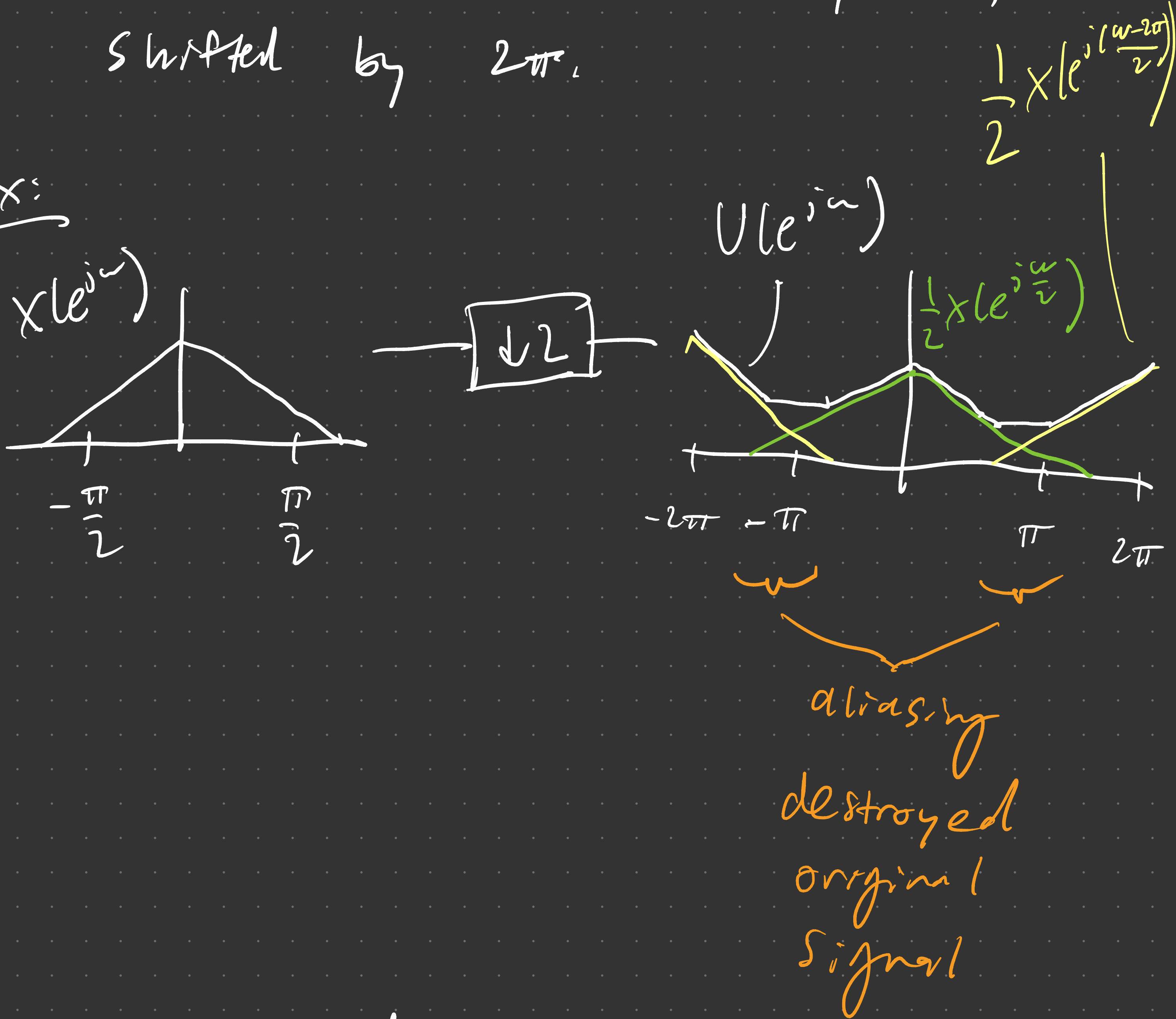
$$\Rightarrow u[n] = \frac{1}{2}\left(x[n] + (-1)^n x[n]\right)$$

$$\Rightarrow U(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + X(e^{j(\omega-\pi)})\right)$$

$$\Rightarrow \boxed{V(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\frac{\omega}{2}}) + X(e^{j\left(\frac{\omega-2\pi}{2}\right)})\right)}$$

Obs: Downsampled spectrum is the avg. of original spectrum stretched by factor 2 and that same spectrum shifted by 2π.

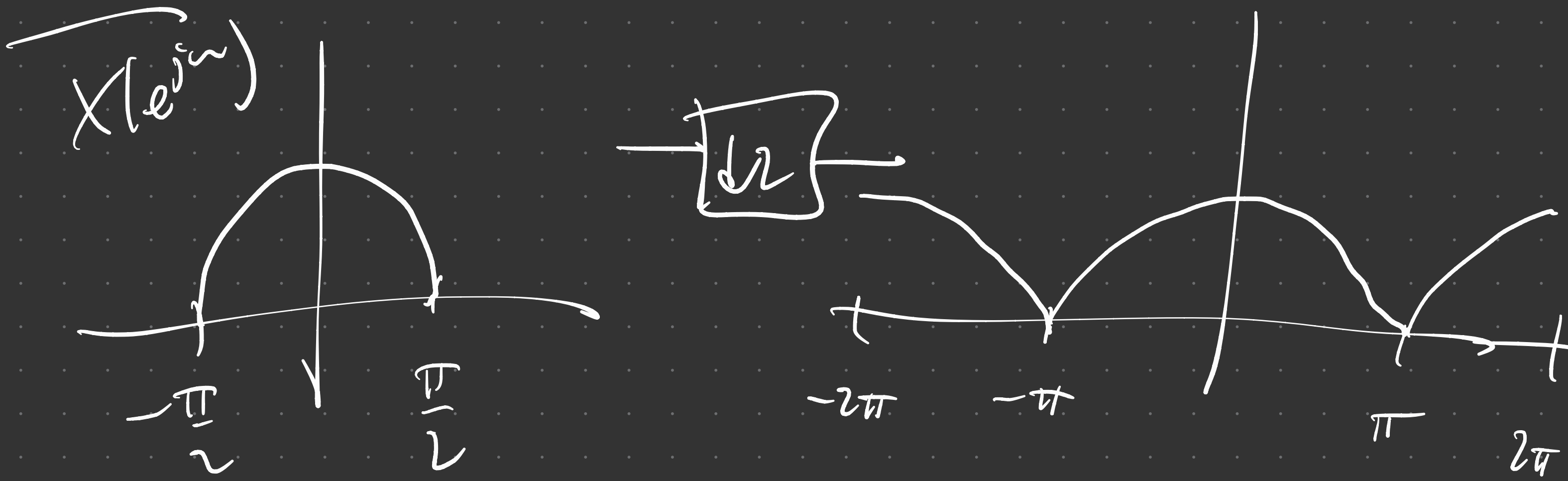
Ex:



Q: What conditions on $X(e^{j\omega})$ for no aliasing?

A: Band limited $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Ex:



Remark: Downsampling is always preceded by filtering

Decimation

antialiasing filter

Obs: Reverse of upsampling.