Last Time: Multirate Operations

Downsampling by Factor 2

$$x[n] \longrightarrow \boxed{\downarrow 2} \xrightarrow{\quad v[n] \quad} \boxed{\uparrow 2} \longrightarrow u[n]$$

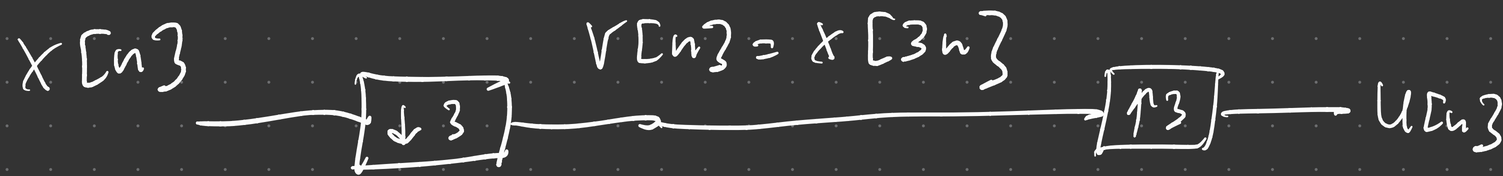
- $U(e^{j\omega}) = V(e^{j2\omega})$

- $u[n] = \frac{1}{2}\left(x[n] + (-1)^n x[n] \right)$

$$\Rightarrow U(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + X(e^{j(\omega-\pi)}) \right)$$

$$\Rightarrow V(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\frac{\omega}{2}}) + X(e^{j\left(\frac{\omega-2\pi}{2}\right)}) \right)$$

Downsampling by factor 3

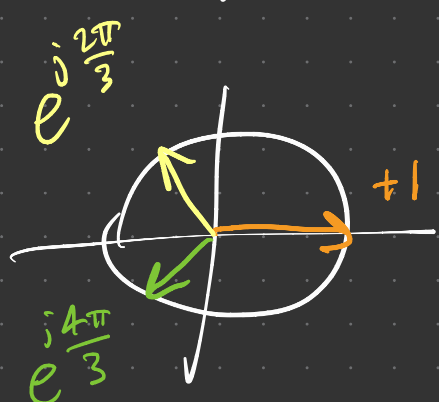


How do we generalize: (↓2)

$$u[n] = \frac{1}{2}\left((+1)^n x[n] + (-1)^n x[n]\right)$$

Obs: ± 1 are the 2nd roots of unity

Claim: 3rd roots of unity



$$\Rightarrow u[n] = \frac{1}{3}\left(x[n] + e^{j\frac{2\pi}{3}n} x[n] + e^{j\frac{4\pi}{3}n} x[n]\right)$$

- $U(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j\left(\omega - \frac{2k\pi}{3}\right)}\right)$

- $U(e^{j\omega}) = V(e^{j3\omega})$

$$\boxed{V(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j\left(\frac{\omega - 2k\pi}{3}\right)}\right)}$$

<u>Remark:</u> Before downsampling by factor 3, band limit signal to $[-\frac{\pi}{3}, \frac{\pi}{3}]$ to avoid aliasing.

Downsample by factor M

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow v[n] = x[Mn]$$

$$\boxed{V(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega - 2k\pi}{M}\right)}\right)}$$

$$\boxed{V(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} e^{-j\frac{2k\pi}{M}}\right)}$$

Exercise: Compare

$$\longrightarrow \boxed{\downarrow M} \longrightarrow \boxed{\uparrow M} \longrightarrow \quad \text{vs.} \quad x[n] \longrightarrow \boxed{\uparrow M} \longrightarrow \boxed{\downarrow M} \longrightarrow x[n]$$

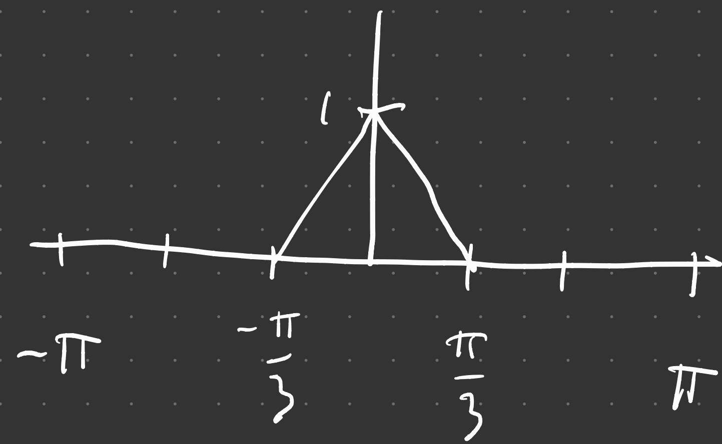
keeps every Mth coeff. and sets the rest to 0.

identity

Ex:

$x[n] \rightarrow \boxed{\downarrow 3} \rightarrow v[n]$

$X(e^{j\omega})$

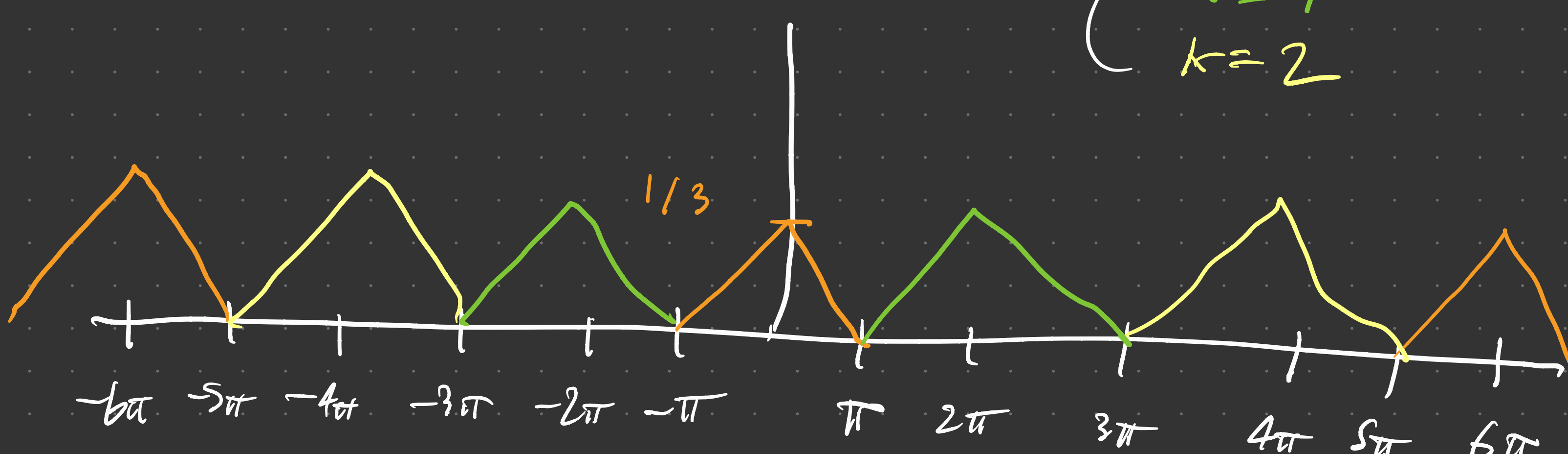


$$V(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j\left(\frac{\omega - 2k\pi}{3}\right)}\right)$$

$\boxed{\text{No aliasing}}$

All together are valid

- $k=0$ valid? No. Not 2π-periodic.
- $k=1$
- $k=2$



Noble Identities

Recall: Multirate DSP is hard because operations don't commute.

- up/down sampling is time-varying

Noble Identities help to quickly analyze multirate systems.

① $$\rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow \;\equiv\; \rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow$$

② $$\rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z^L)} \rightarrow \;\equiv\; \rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow L} \rightarrow$$

③ If $\gcd\{M, L\}$:

Ex: • $M=3$, $L=4$ ✓
• $M=2$, $L=4$ ✗

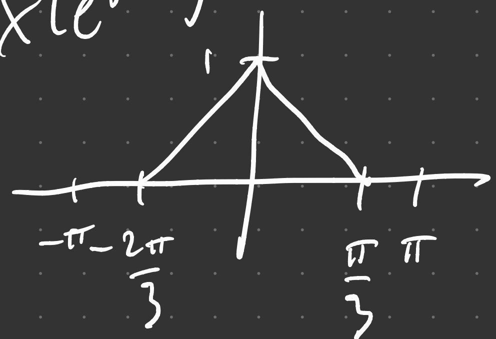
$$\rightarrow \boxed{\downarrow M} \rightarrow \boxed{\uparrow L} \rightarrow \;\equiv\; \rightarrow \boxed{\uparrow L} \rightarrow \boxed{\downarrow M} \rightarrow$$

Proof: Do it at home (Ch. 3, sec. 4).

Exercise:

$x[n] \to \boxed{z^{-3}} \to \boxed{\uparrow 2} \to \boxed{\downarrow 3} \to \boxed{\uparrow 3} \to y[n]$

$X(e^{j\omega})$



What is $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$?

Soln: Don't do it with brute force!

$X(z) \to \boxed{z^{-3}} \to z^{-3}X(z)$ annoying to deal with...

By 3: $\to \boxed{z^{-3}} \to \boxed{\downarrow 3} \to \boxed{\uparrow 2} \to \boxed{\uparrow 3} \to$

By 1: $\to \boxed{\downarrow 3} \to \boxed{z^{-1}} \to \boxed{\uparrow 2} \to \boxed{\uparrow 3} \to$

By 2: $\to \boxed{\downarrow 3} \to \boxed{\uparrow 2} \to \boxed{z^{-2}} \to \boxed{\uparrow 3} \to$

By 2: $x[n] \to \boxed{\downarrow 3} \to \boxed{\uparrow 2} \to \boxed{\uparrow 3} \xrightarrow{v[n]} \boxed{z^{-6}} \to y[n] = v[n-6]$

$$x[n] \to \boxed{\downarrow 3} \to \boxed{\uparrow 3} \xrightarrow{u[n]} \boxed{\uparrow 2} \to v[n]$$

$$U(z) = \frac{1}{3} \sum_{k=0}^{2} X\left(z\, e^{-j\frac{2k\pi}{3}}\right)$$

$$V(z) = \frac{1}{3} \sum_{k=0}^{2} X\left(z^2 e^{-j\frac{2k\pi}{3}}\right)$$

$$V(e^{j\omega}) = \frac{1}{3} \sum_{k=0}^{2} X\left(e^{j2\left(\omega - \frac{k\pi}{3}\right)}\right)$$

$$Y(e^{j\omega}) = e^{-j6\omega} \cdot \frac{1}{3} \sum_{k=0}^{2} X\left(e^{j2\left(\omega - \frac{k\pi}{3}\right)}\right)$$

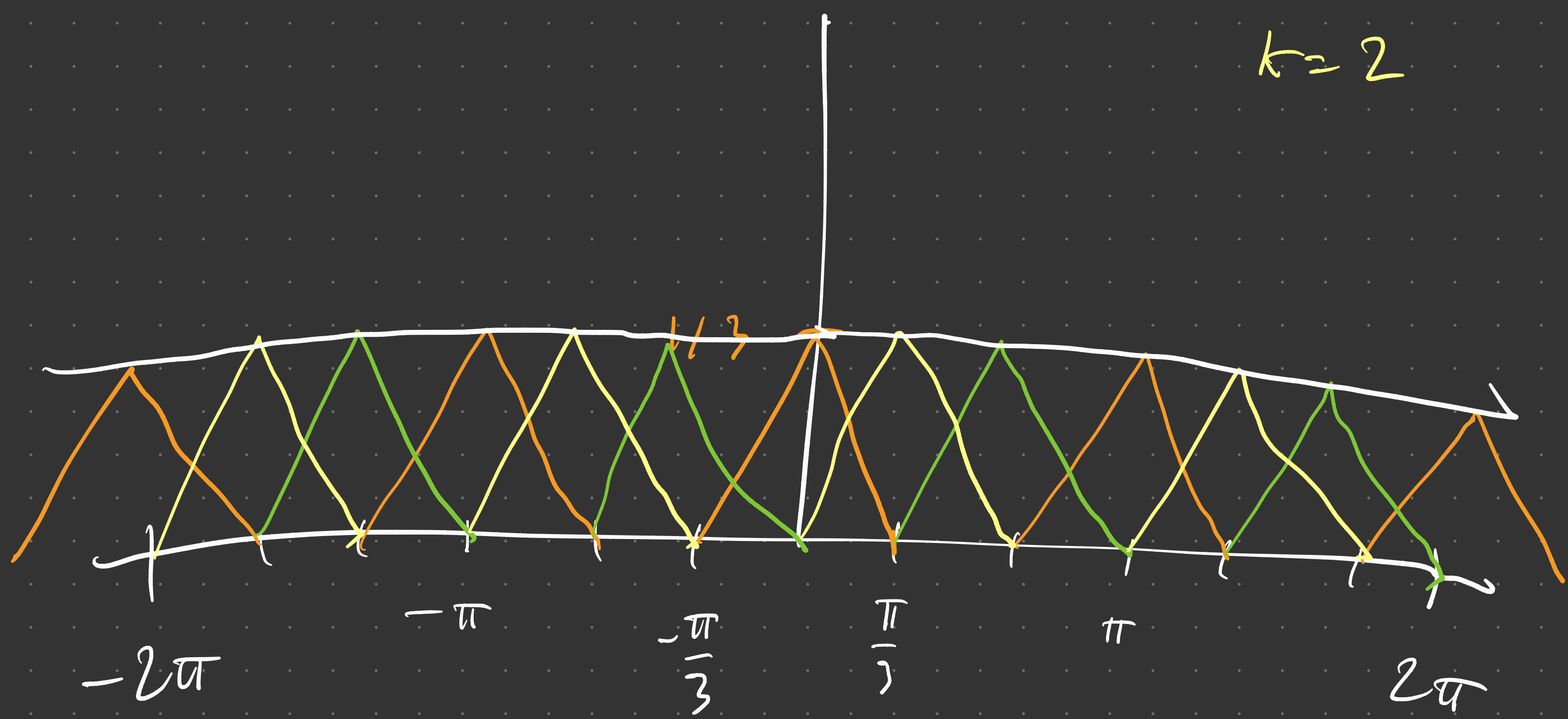
<u>Exer:</u> Try different orders of the Noble identities.

$|Y(e^{j\omega})|$

$k=0$

$k=1$

$k=2$



Obs: Severe aliasing?

Q: How could we see this before?

A: Input NOT bandlimited to $[-\frac{\pi}{3}, \frac{\pi}{3}]$ and we immediately downsample by 3.

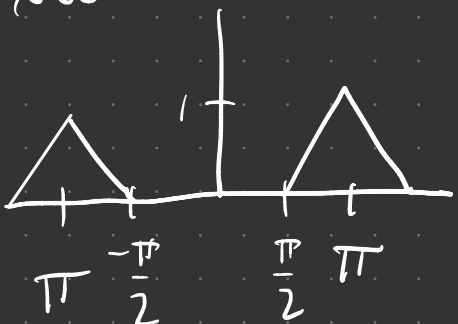
Exercise:

$H(e^{j\omega})$

ideal low-pass filter

$$x[n] \rightarrow \boxed{\uparrow 2} \rightarrow \boxed{\downarrow 4} \rightarrow \boxed{\uparrow 4} \rightarrow \boxed{\downarrow 2} \rightarrow \boxed{H} \rightarrow y[n]$$

$X(e^{j\omega})$



What is $Y(e^{j\omega})$ in terms of $X(e^{j\omega})$?

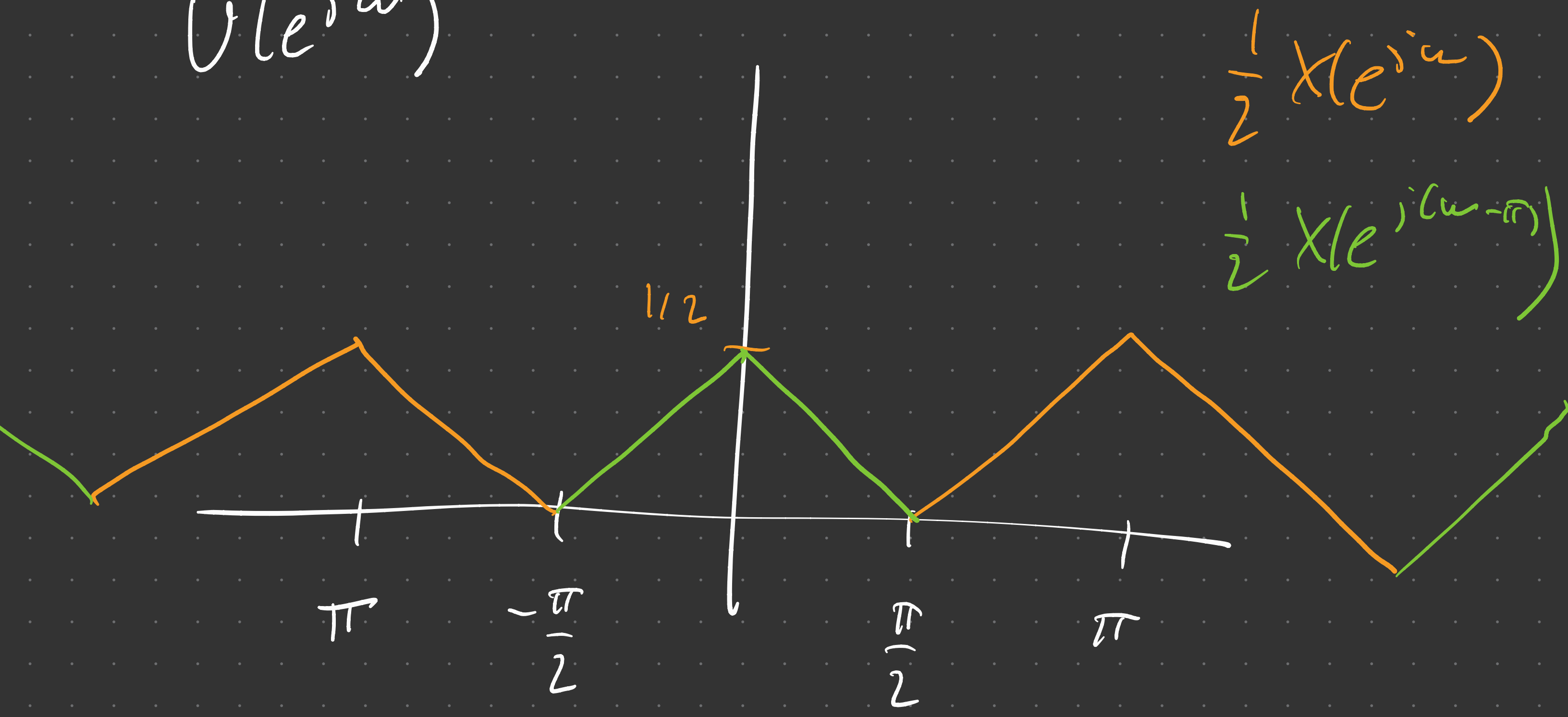
Sol$^{\underline{n}}$:

$$\rightarrow \underbrace{\boxed{\uparrow 2} \rightarrow \boxed{\downarrow 2}}_{\text{identity}} \rightarrow \boxed{\downarrow 2} \rightarrow \boxed{\uparrow 2} \rightarrow \underbrace{\boxed{\uparrow 2} \rightarrow \boxed{\downarrow 2}}_{\text{identity}} \rightarrow \boxed{H} \rightarrow$$

$$x[n] \rightarrow \boxed{\downarrow 2} \rightarrow \boxed{\uparrow 2} \xrightarrow{u[n]} \boxed{H} \rightarrow y[n]$$

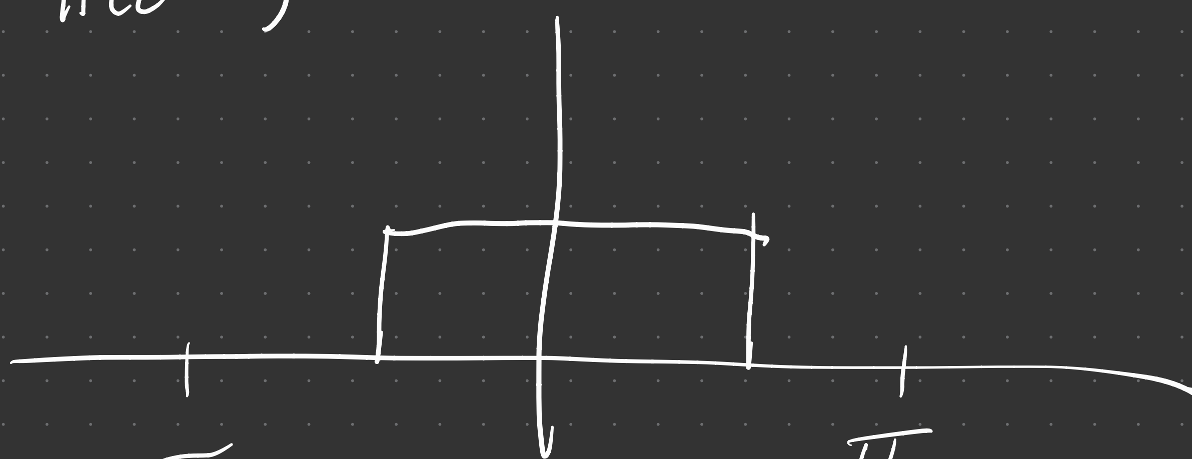
$$U(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + X(e^{j(\omega-\pi)})\right)$$

$U(e^{j\omega})$



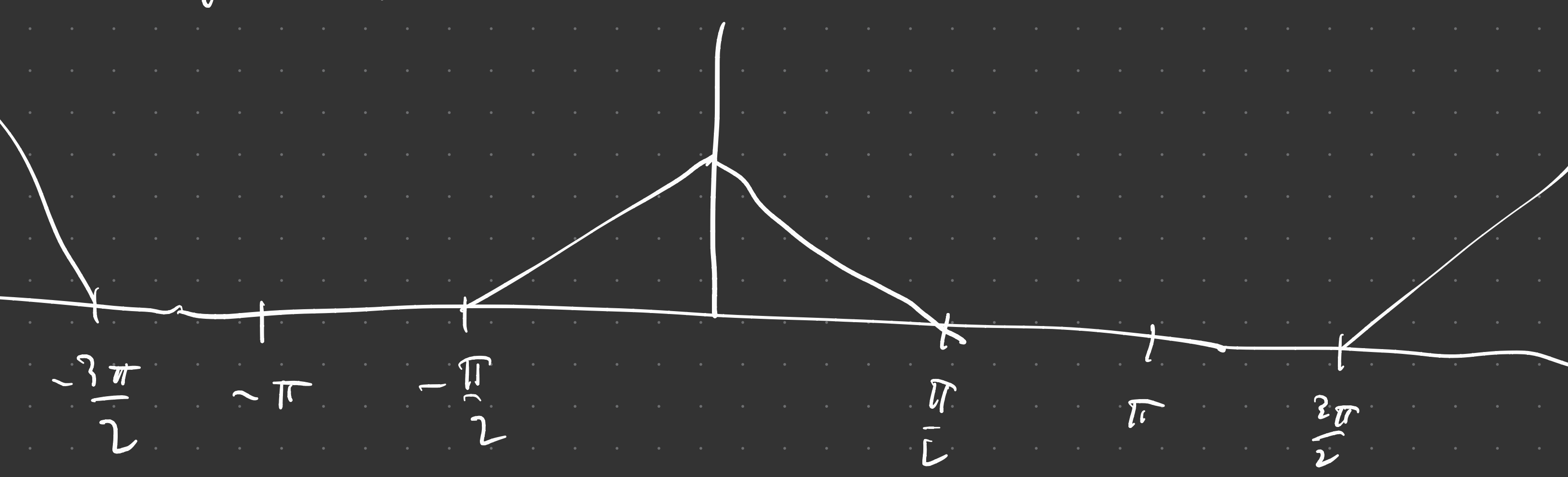
$\cdot$

$H(e^{j\omega})$



$=$

$Y(e^{j\omega})$



Polyphase Representation

Recall:

$$\rightarrow \boxed{\uparrow L} \rightarrow \boxed{F} \rightarrow$$

$$\rightarrow \boxed{H} \rightarrow \boxed{\downarrow M} \rightarrow$$

This is only for theory. Extremely wasteful in practice.

Thought Experiment:

Imagine $L = M = 1024$.

- F is processing a lot of zeros
  - Expensive
- Throwing away most of the computations after H.

Q: Can we filter before upsampling?

Q: Can we filter after downsampling?

Ex: $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

$$= \underbrace{(1 + 3z^{-2})}_{\text{even powers}} + \underbrace{(2z^{-1} + 4z^{-3})}_{\text{odd powers}}$$

$$= (1 + 3z^{-2}) + z^{-1}(2 + 4z^{-2})$$

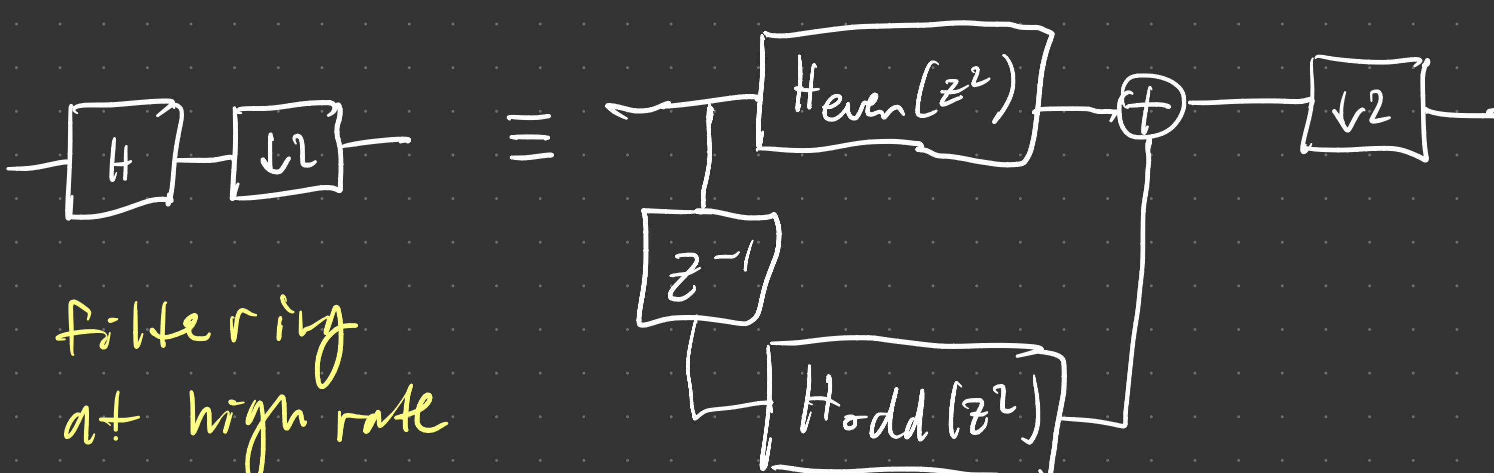
$$= H_{even}(z^2) + z^{-1} H_{odd}(z^2),$$

where

$$H_{even}(z) = (1 + 3z^{-1})$$

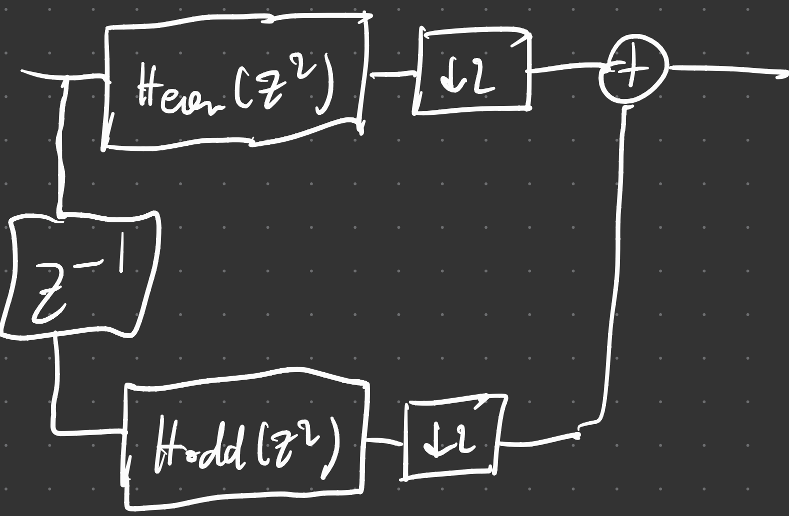
$$H_{odd}(z) = (2 + 4z^{-1})$$

These are the even/odd polyphases.

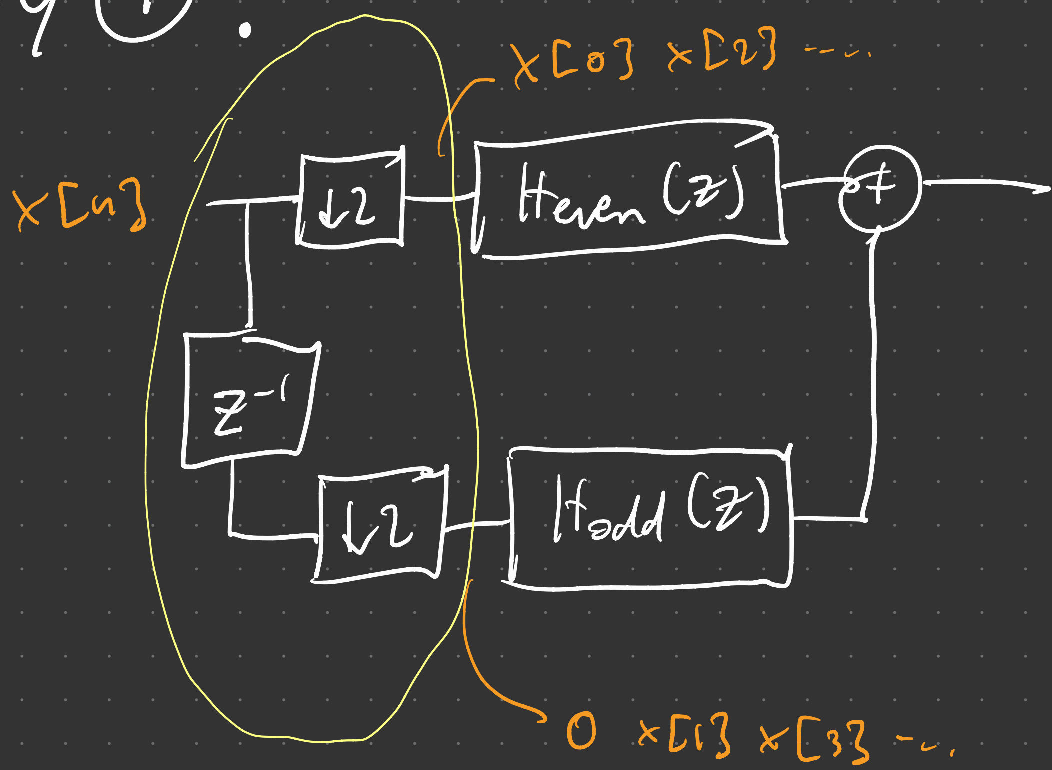


filtering at high rate = slow

Since downsampling is linear:



By (1):



filtering at low rate = fast

Series-to-parallel buffer

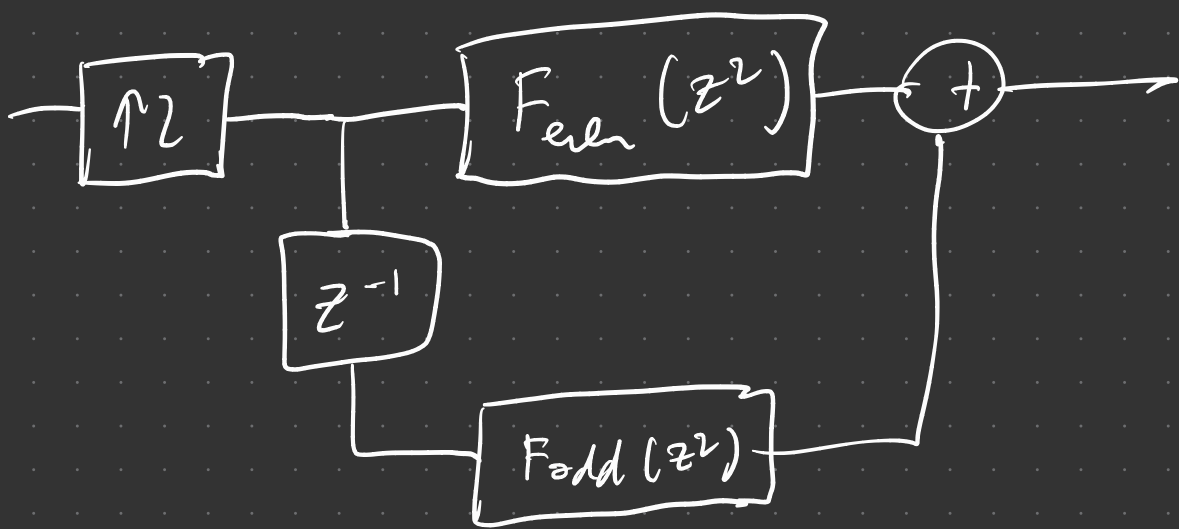
Remark: This is how every multirate system is implemented.

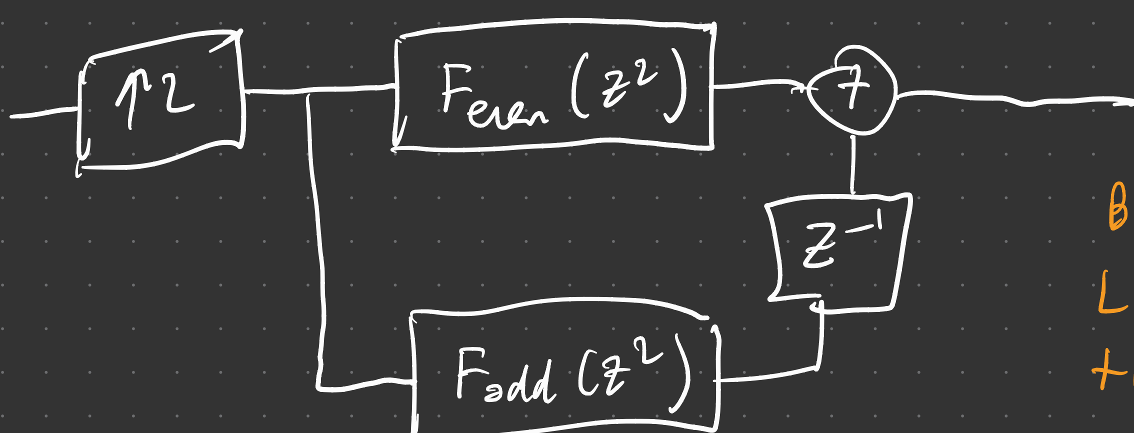
Q: Why is this useful?

Exercise :

slow

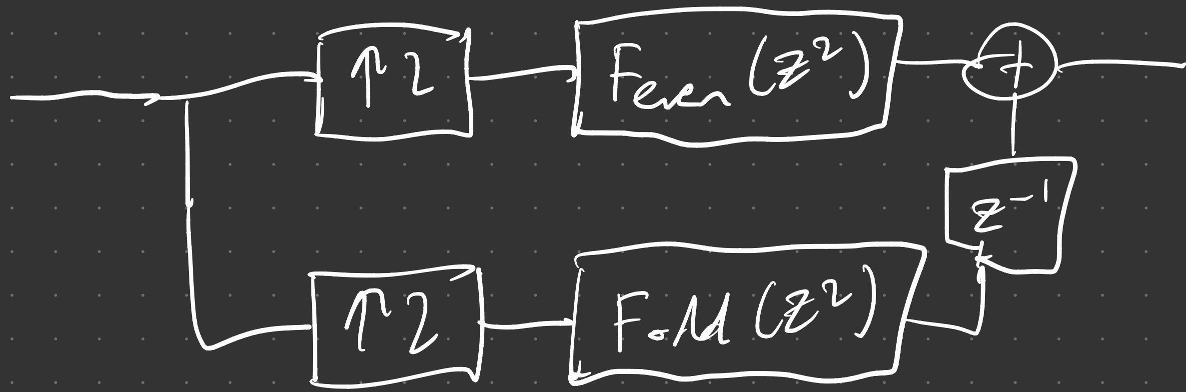
$$F(z) = F_{even}(z^2) + z^{-1} F_{odd}(z^2)$$



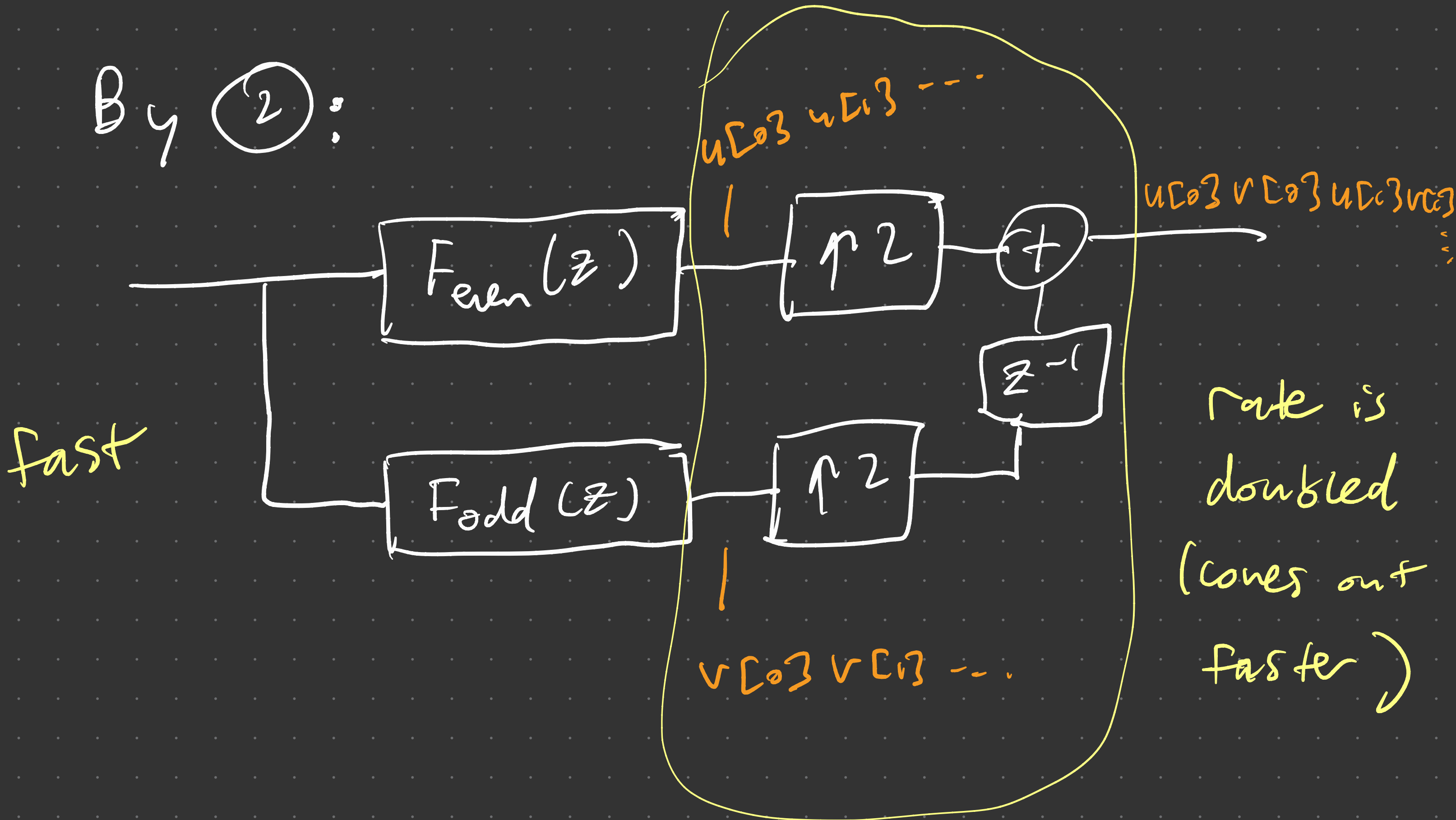


Both are LTI so they commute.

Since upsampling is linear:



By ②:



multiplexing

parallel-to-serial buffer

Remark: Polyphase representations efficiently implement multirate operations.

Obs: for FIR filters its very clear.

Q: What about IIR filters?

Exercise:

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

What are the even and odd polyphases?

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \underbrace{\frac{1+\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}}}_{1}$$

$$= \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}}$$

$$= \underbrace{\left(\frac{1}{1-\frac{1}{4}z^{-2}}\right)}_{\text{even}} + z^{-1}\underbrace{\left(\frac{\frac{1}{2}}{1-\frac{1}{4}z^{-2}}\right)}_{\text{odd}}$$

$$H_{even}(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}, \quad H_{odd}(z) = \frac{\frac{1}{2}}{1 - \frac{1}{4}z^{-1}}$$

Obs: The even/odd polyphases of an IIR filter are both IIR.

Time-Domain Characterization

$$h[n]$$

$$h_{even}[n] = h[2n] \qquad\qquad h_{odd}[n] = h[2n+1]$$

Ex:

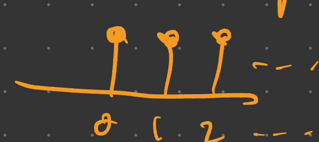
$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$h[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

$$H(z) = \frac{1}{1 - az^{-1}}$$

$$h[n] = a^n u[n]$$

unit step



$$h_{even}[n] = \left(\frac{1}{2}\right)^{2n} u[2n] = \left(\frac{1}{4}\right)^{n} u[n]$$

$$H_{even}(z) = \frac{1}{1 - \frac{1}{4} z^{-1}}$$

$$h_{odd}[n] = \left(\frac{1}{2}\right)^{2n+1} u[2n+1] = \frac{1}{2}\left(\frac{1}{4}\right)^{n} u[n]$$

$$H_{odd}(z) = \frac{\frac{1}{2}}{1 - \frac{1}{4} z^{-1}}$$