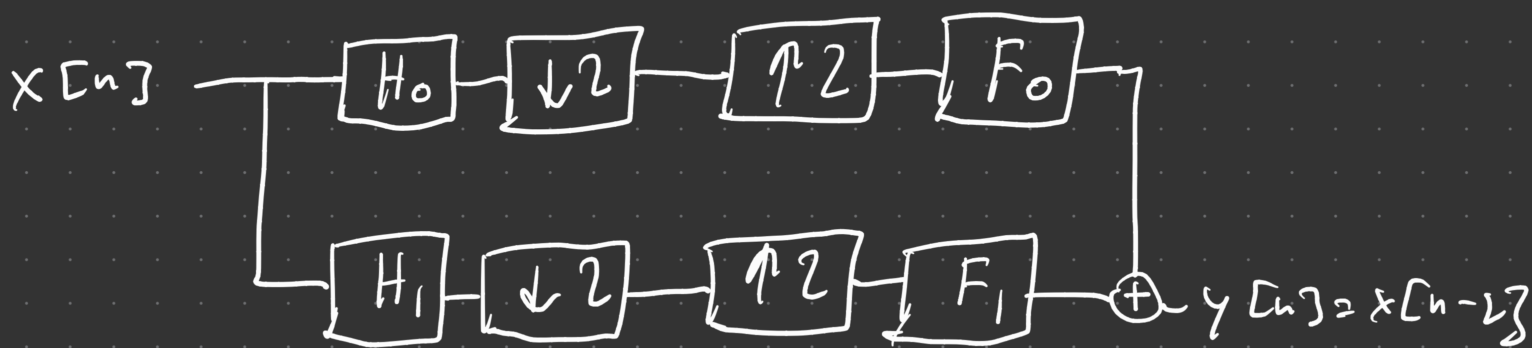
Last Time: PR Filter Banks

Design procedure for

- Perfect-Reconstruction
- FIR
- Two-Channel

filter banks.



<u>PR Conditions</u>

- $F_0(z)\,H_0(z) + F_1(z)\,H_1(z) = 2z^{-L}$
- $F_0(z)\,H_0(-z) + F_1(z)\,H_1(-z) = 0$

Design Procedure

① Design a half-band Type I linear-phase filter P_0 with center of symmetry

$$P_0(z) - P_0(-z) = 2z^{-L}, \quad L \text{ is an odd integer}$$

② Factorize $P_0(z)$ into $H_0(z)$ and $F_0(z)$

③ Define $H_1(z) = F_0(-z)$

$$F_1(z) = -H_0(-z)$$

Recall: $P_0(e^{j\omega}) = e^{-jL\omega} \underbrace{P_{ampl}(\omega)}_{\text{real function}}$

center of symmetry

$$\boxed{P_{ampl}(\omega) + P_{ampl}(\omega - \pi) = 2}$$

- $\omega_p + \omega_s = \pi$
- ripples must cancel out $(\delta_p = \delta_s)$

Last time we showed that

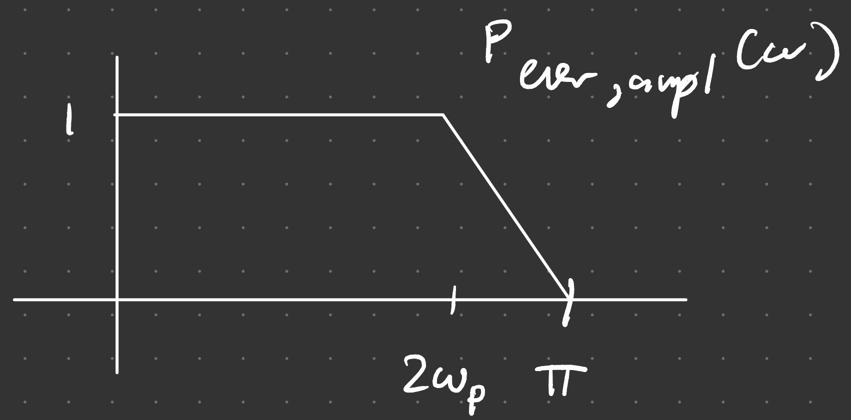
$$P_0(z) = P_{0,even}(z^2) + z^{-L}$$

odd polyphase **must** be a delay

and that $P_{0,even}$ is a Type II linear-phase filter with center of symmetry $\frac{L}{2}$.

$$P_{0,even}(e^{j\omega}) = e^{-j\frac{L}{2}\omega} P_{even,ampl}(\omega)$$

We saw that $P_{even,ampl}(\omega)$ is a one-band filter



Cartoon diagram

(ECE 161 A/B/C) MATLAB: fir2

Remark: We are experts at step 1 in the design procedure.

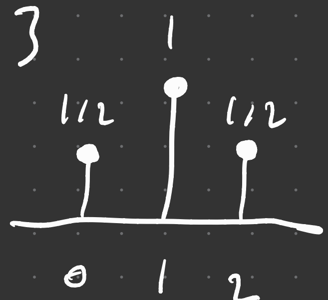
Step ②: Given $P_0(z)$, how do we distribute its zeros across $H_0(z)$ and $F_0(z)$?

Ex: What's the simplest half-band filter?

$$P_0(z) = \frac{1}{2}\left(1+z^{-1}\right)^2$$

$$= \frac{1}{2}\left(1+2z^{-1}+z^{-2}\right) = \frac{1}{2} + z^{-1} + \frac{1}{2}z^{-2}$$

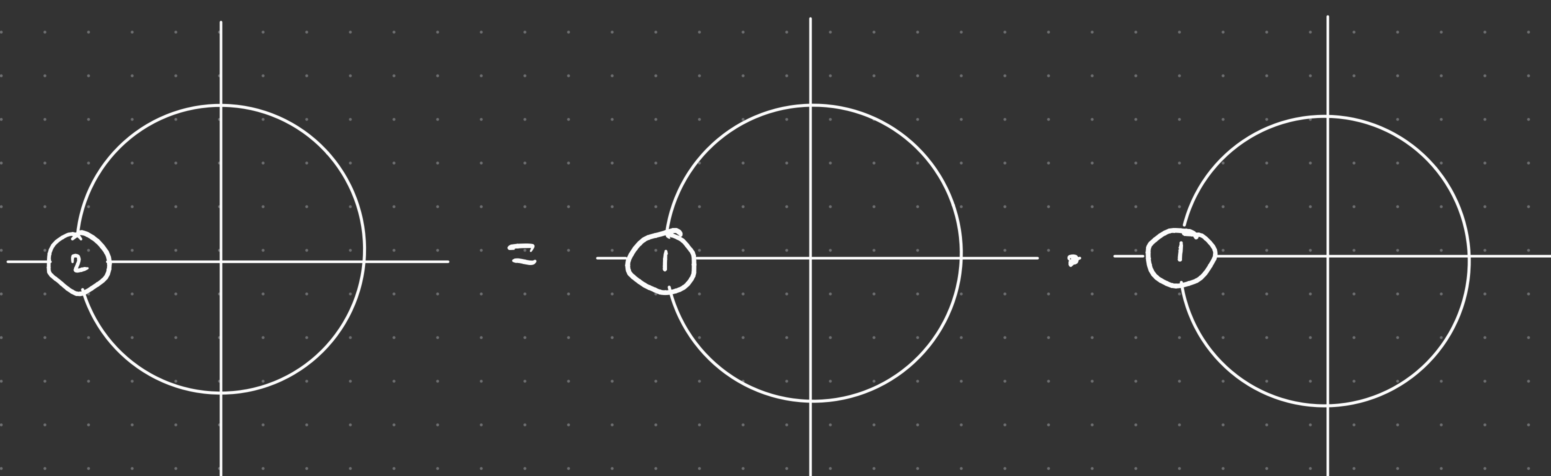
$p_0[n]$



2nd-order interp. filter

Type I linear-phase

$$P_0(z) \quad = \quad F_0(z) \quad \cdot \quad H_0(z)$$



$$\frac{1}{2}\left(1+z^{-1}\right)^2 \qquad \frac{1}{\sqrt{2}}\left(1+z^{-1}\right) \qquad \frac{1}{\sqrt{2}}\left(1+z^{-1}\right)$$

Obs:

- $F_0(z) = H_0(z)$
- F_0 & H_0 are of the same order ($=1$)
- $F_0(z) = z^{-1} H_0(z^{-1})$

$\longrightarrow$ orthogonal filter bank

$\longrightarrow$ orthogonal wavelets

- $H_1(z) = F_0(-z)$

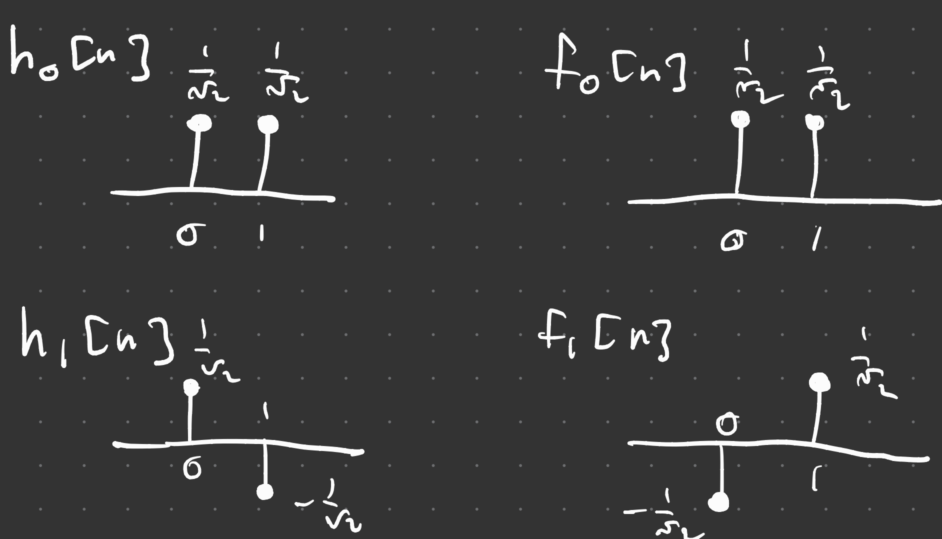
$$H_1(e^{j\omega}) = F_0\left(e^{j(\omega - \pi)}\right)$$ shift in frequency

$$h_1[n] = e^{j\pi n} f_0[n]$$ modulation in time

$$= (-1)^n f_0[n]$$ flip odd coeffs

- $F_1(z) = -H_0(-z)$

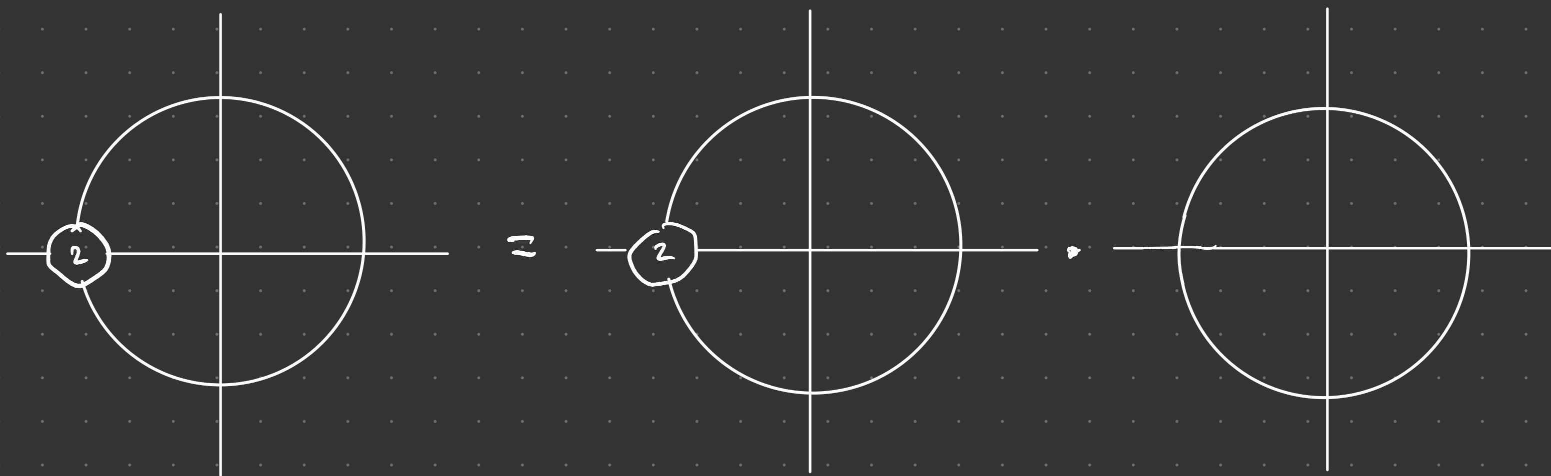
$$f_1[n] = (-1)^{n+1} h_0[n]$$ flip even coeffs



$\boxed{\text{Haar wavelet filters} \quad db(1) \quad \text{Daubechies}}$

Q: Are there other possible factorizations?

$$P_0(z) = F_0(z) \cdot H_0(z)$$



$$\frac{1}{2}(1+z^{-1})^2 \qquad \frac{1}{2}(1+z^{-1})^2 \qquad 1$$

or flip these

Q: What is the next possible order?

A: 6th-order

Obs: General rule is to add 4 to get the next valid order.

6th-order:

Remark: There are many different possible design choices.

Q: How do we get an orthogonal filter bank/wavelets?

A: $F_0(z) = z^{-3} H_0(z^{-1})$

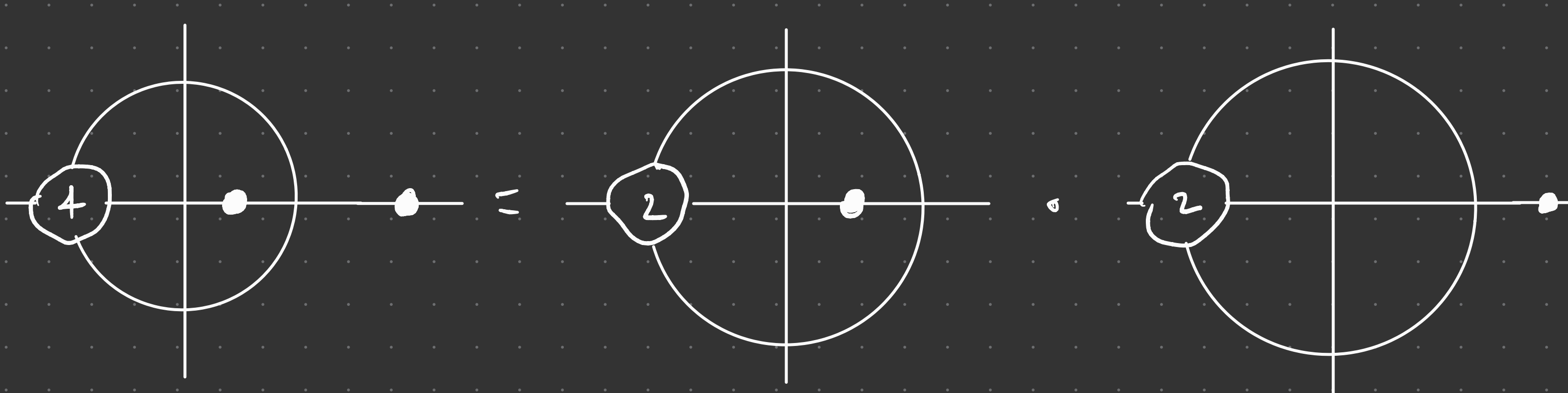
General: Orthogonality is guaranteed when

$$F_0(z) = z^{-L} H_0(z^{-1})$$ flipped zeros

Obs: F_0 and H_0 have the same order.

Ex: $\boxed{\text{db(2) wavelet filters}}$

$$P_0(z) = F_0(z) \cdot H_0(z)$$



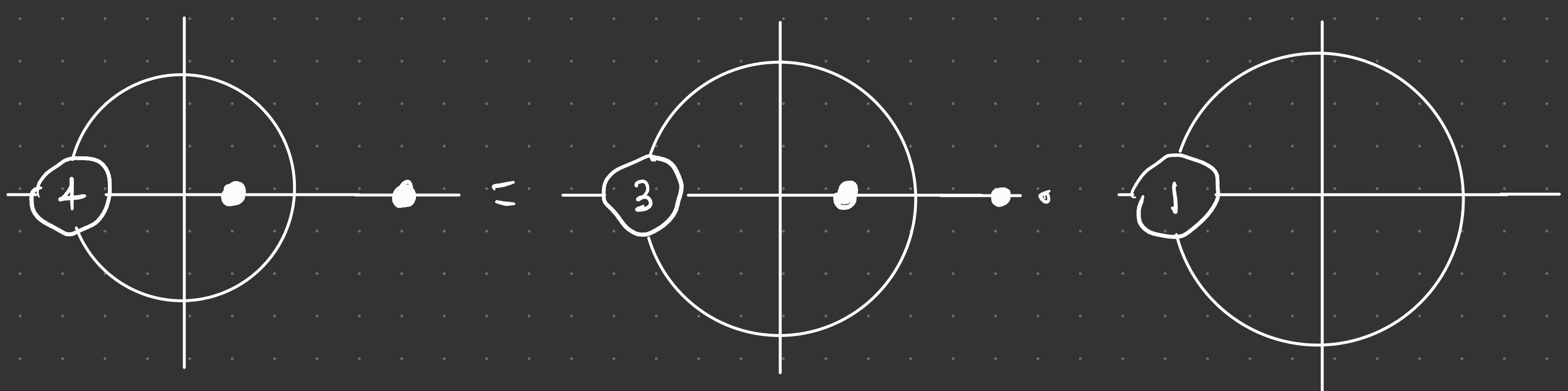
Type I linear-phase   not linear-phase   not linear-phase

Remark: Orthogonal filter banks cannot be linear-phase except for the Haar.

Note: In the example above, even though F_0 & H_0 are not linear-phase, the product filter $P_0(z) = F_0(z) H_0(z)$ is linear-phase.

Ex: $\boxed{\text{bior}(3,1)}$

$$P_0(z) = F_0(z) \cdot H_0(z)$$

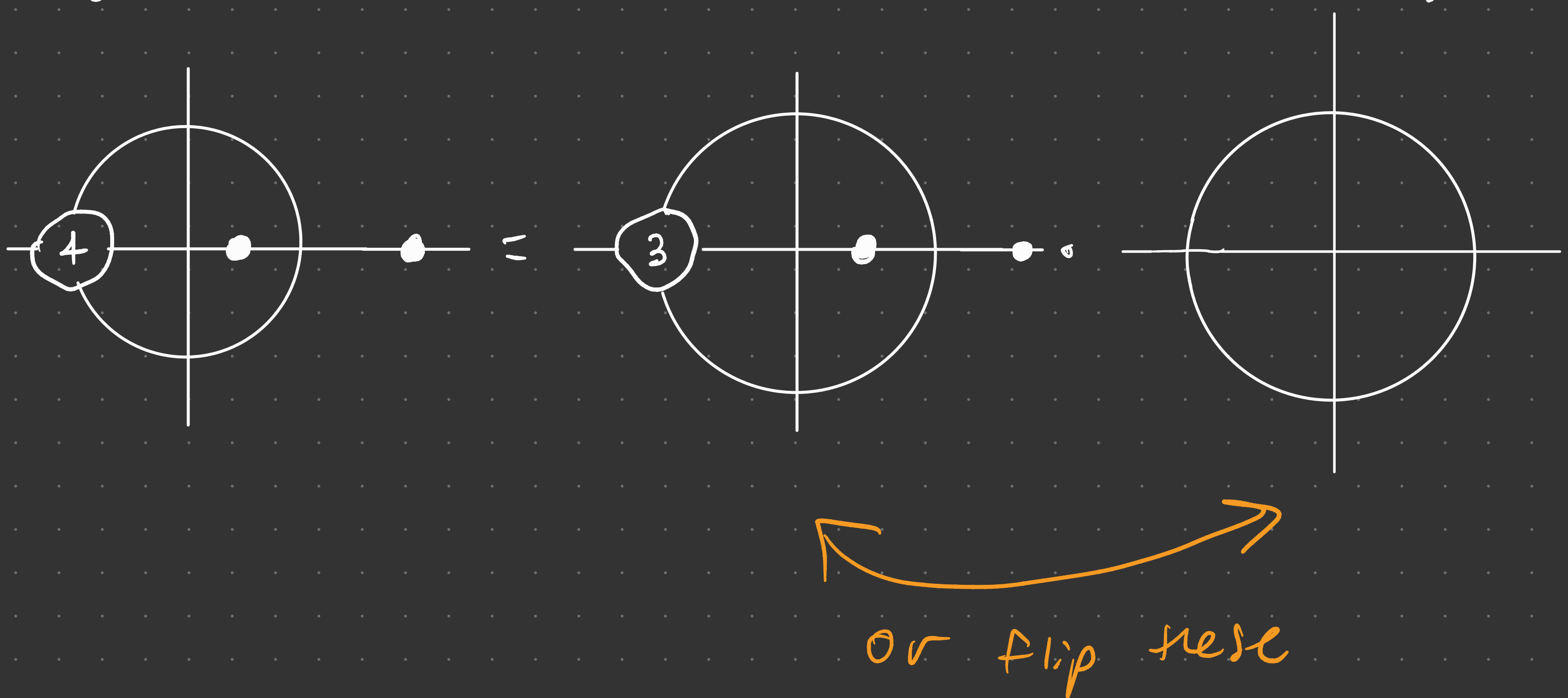


Type I linear-phase   Type II linear-phase   Type II linear-phase

Remark: This filter bank is called biorthogonal.

Ex:

$$P_0(z) = F_0(z) \cdot H_0(z)$$



Remark: This kind of system is not very useful. The general rule of thumb is to balance the zeros as much as possible.

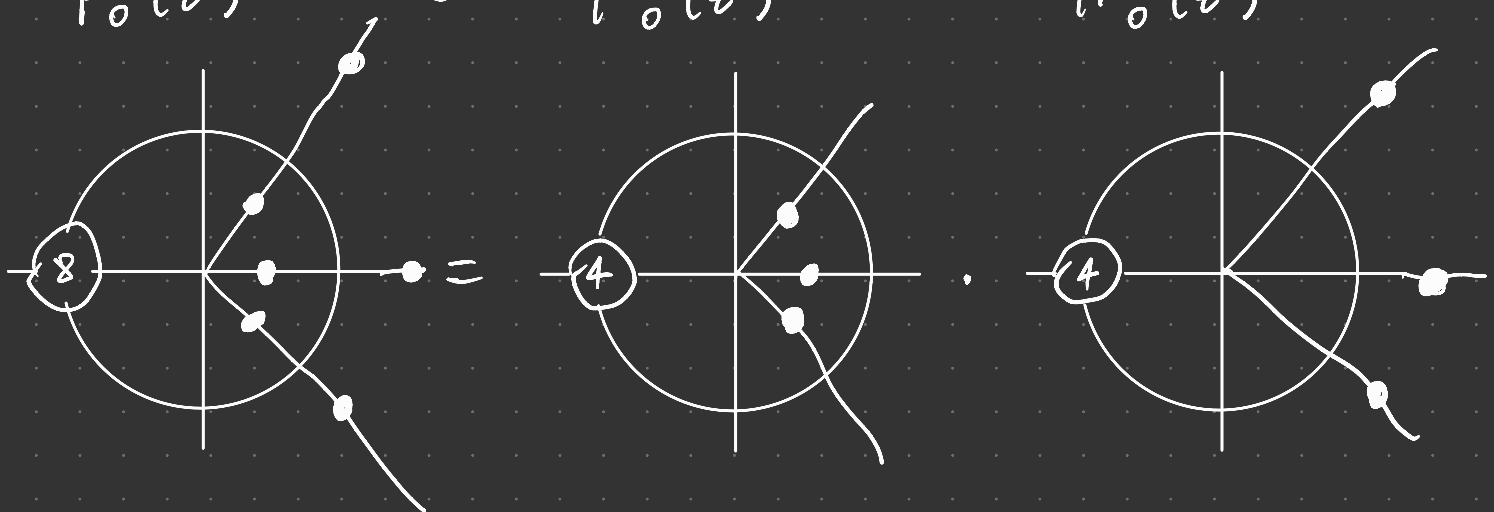
14th - order

To guarantee orthogonality, we want

$$F_0(z) = z^{-7} H_0(z^{-1})$$

Ex: $\boxed{db(4)}$

$P_0(z) = F_0(z) \cdot H_0(z)$



Type I linear-phase

not linear-phase
min-phase

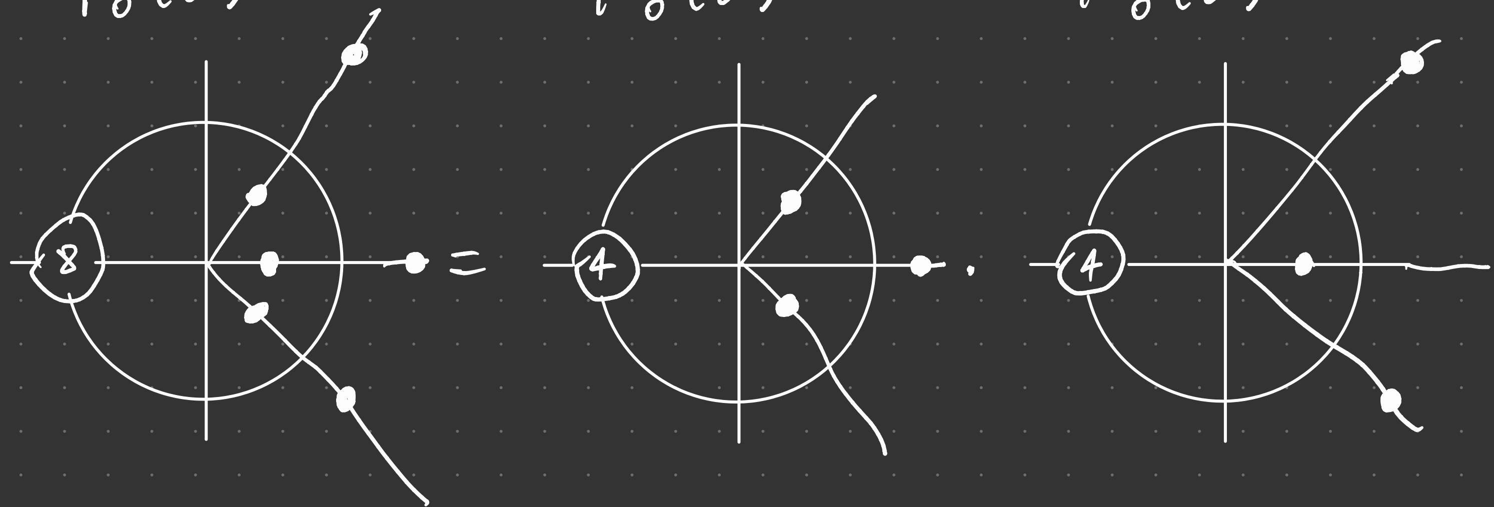
not linear-phase
max-phase

Obs: This factorization produces an orthogonal filter bank/wavelet system.

Obs: This factorization is <u>not</u> unique.

Ex: $\boxed{\text{sym}(4)}$

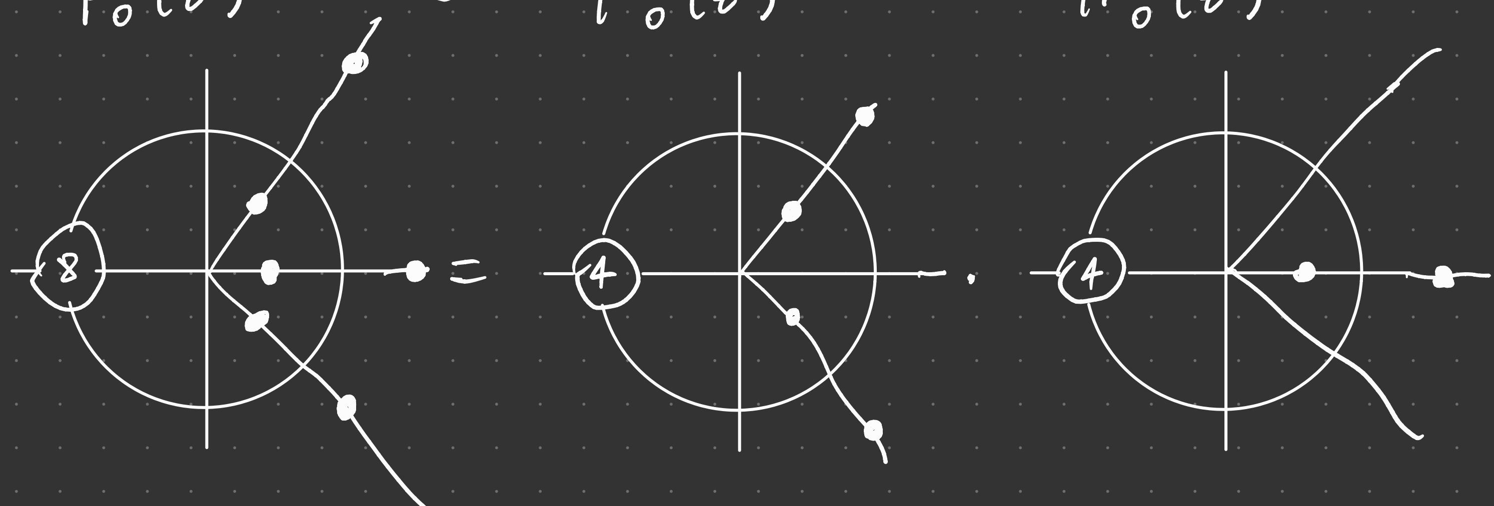
$$P_0(z) = F_0(z) \cdot H_0(z)$$



Obs: This factorization produces an orthogonal filter bank/wavelet system.

Ex: $\boxed{\text{bior}(4,4)}$ JPEG2000

$$P_0(z) = F_0(z) \cdot H_0(z)$$



Type I linear-phase Type I linear-phase Type I linear-phase

Obs: This factorization produces a biorthogonal FB.

Remark: More zeros @ π, the "smoother" the resulting filters are.

- Smoother wavelets
- Better signal-approximation properties.

Goal: People try to design filters with the maximum # of zeros @ π.

- Max-flat filters.

At this point we are "experts" at the design procedure for

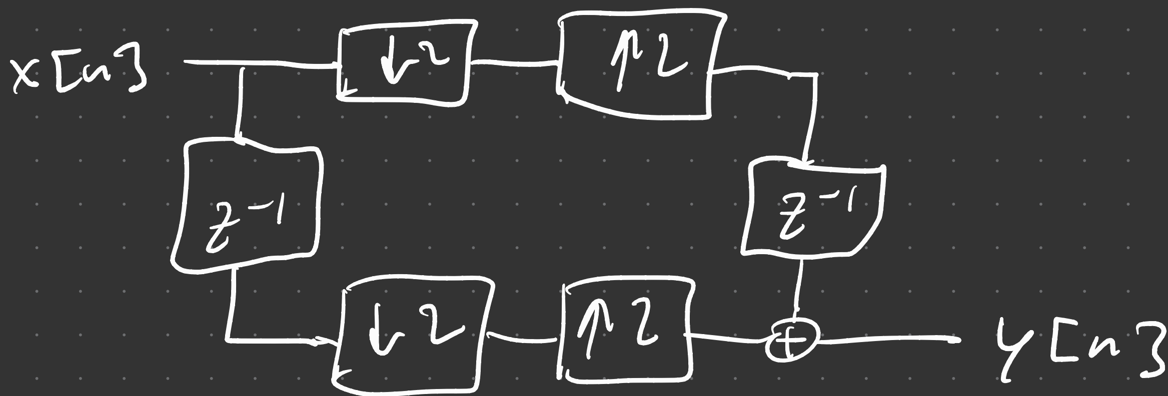
- Perfect-Reconstruction
- FIR
- Two-Channel

filter banks with linear-phase product filters.

- Designing (bi)orthogonal wavelets

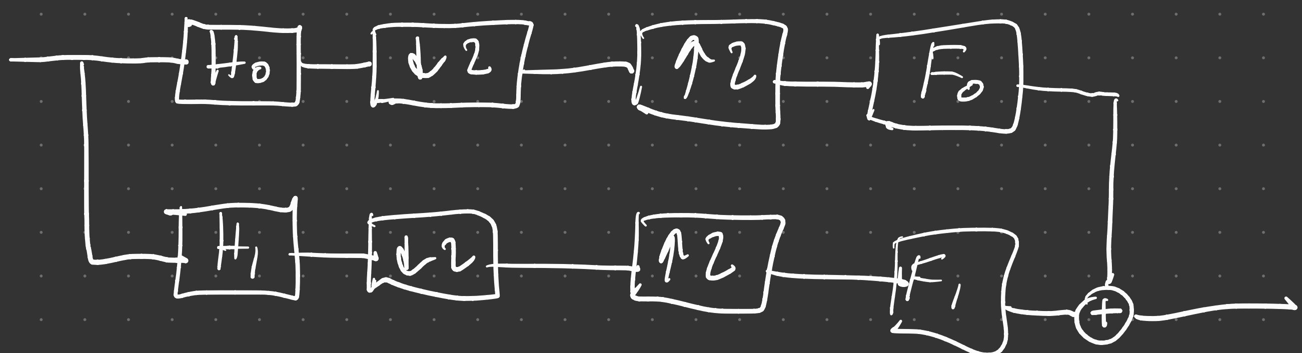
Polyphase Representation of Filter Banks

Exercise:



Show that $y[n] = x[n-1]$.

Consider:



Goal: Write down all of these filters in their respective polyphase representations.

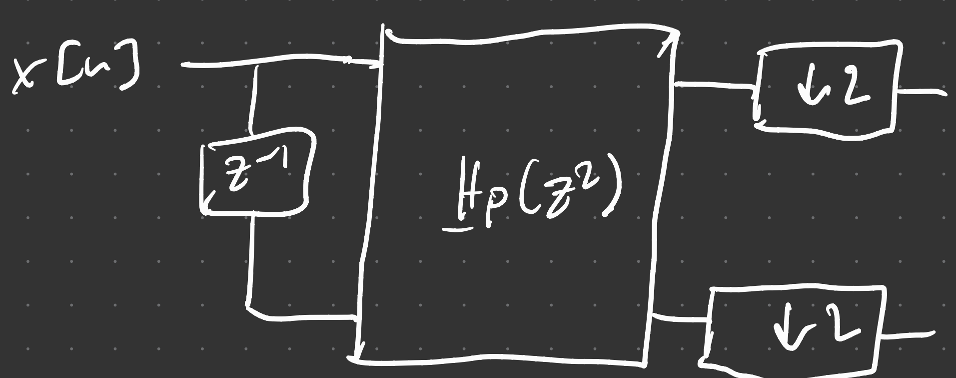
Analysis Bank:

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} H_{0,even}(z^2) + z^{-1} H_{0,odd}(z^2) \\ H_{1,even}(z^2) + z^{-1} H_{1,odd}(z^2) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} H_{0,even}(z^2) & H_{0,odd}(z^2) \\ H_{1,even}(z^2) & H_{1,odd}(z^2) \end{bmatrix}}_{\text{polyphase matrix}} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

$\underline{H}_p(z^2)$

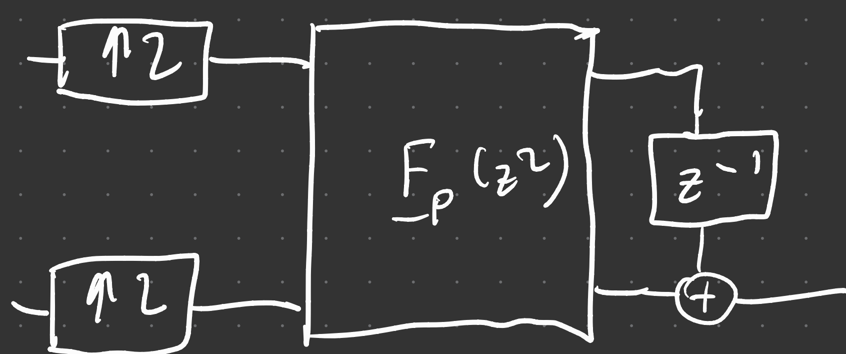
The analysis bank is equivalent to:



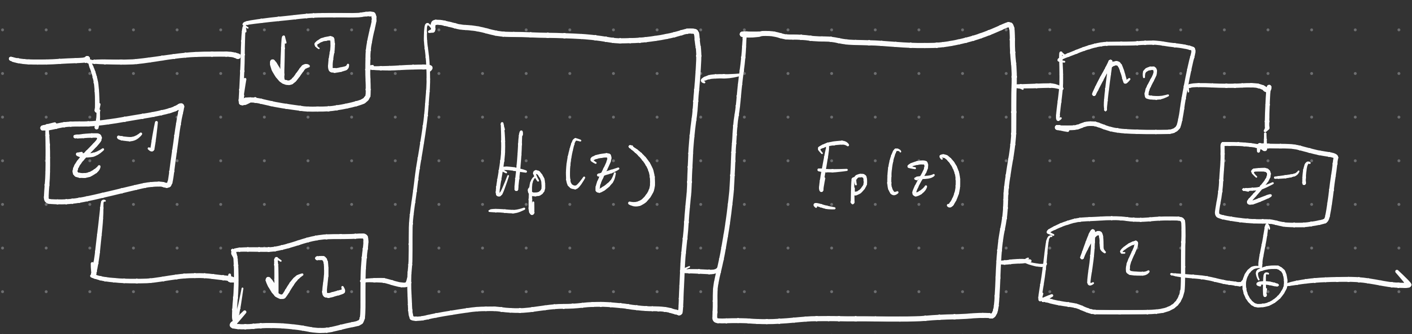
Syntesis Bank:

$$\begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} F_{0,odd}(z^2) & F_{1,odd}(z^2) \\ F_{0,even}(z^2) & F_{1,even}(z^2) \end{bmatrix}}_{\text{polyphase matrix } \underline{F}_p(z^2)}$$

The syntesis bank is equivalent to:



By applying the Noble identities, we find the equivalent system



polyphase representation of the two-channel filter bank.

Obs: Perfect-reconstruction is guaranteed by:

$$\underline{F}_p(z)\,\underline{H}_p(z) = z^{-k} I$$

identity matrix

Exercise: Verify that this is true and determine L.

Q: Are there other conditions that guarantee PR?