Last Time: Polyphase Representations of PR FBs

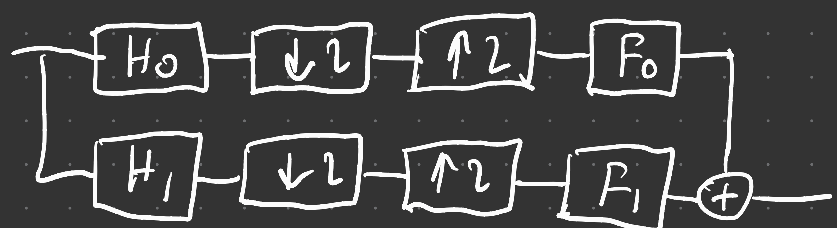
Analysis bank polyphase representation:

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \underline{H}_p(z^2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

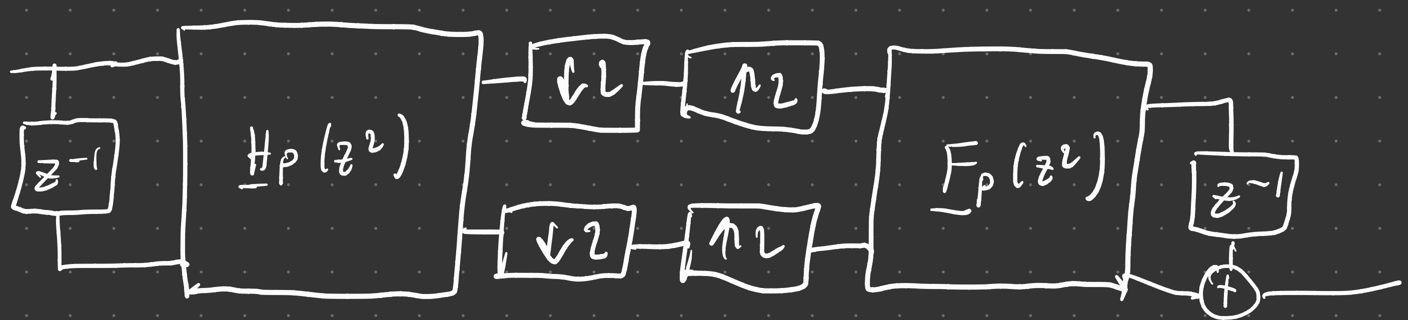
Synthesis bank polyphase representation:

$$\begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \end{bmatrix} \underline{F}_p(z^2)$$

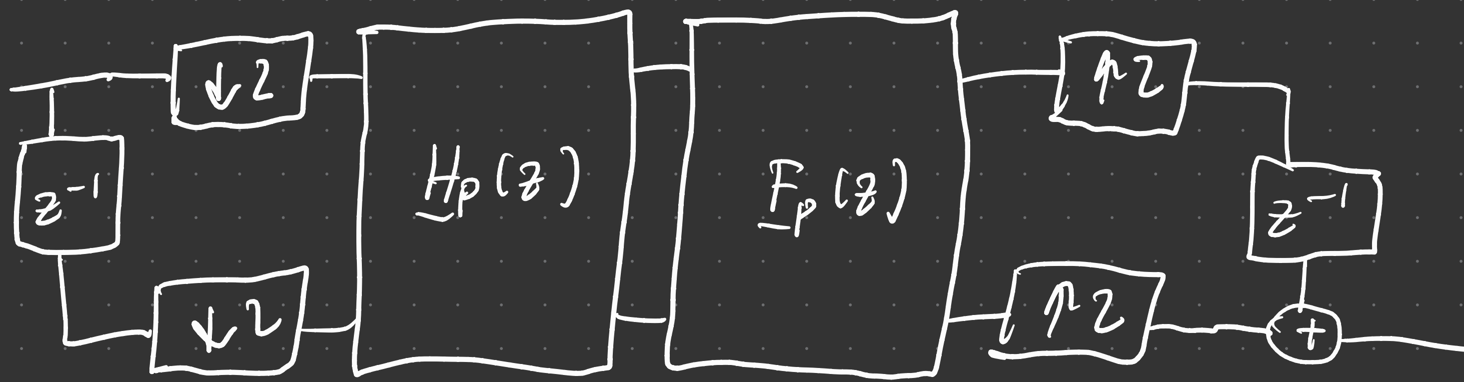
Given a PR filter bank



we have the equivalent system



By the Noble identities, this system is equivalent to



polyphase representation of the two-channel filter bank

If $\underline{F}_p(z)\,\underline{H}_p(z) = z^{-K}\,\mathbb{I}$, identity matrix

then $y[n] = x[n-L]$, where $L = 2K+1$.

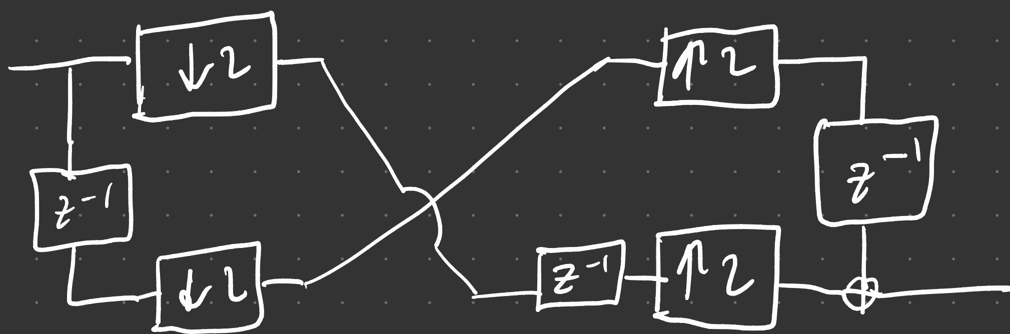
PR

Alternatively, if $\underline{F}_p(z)\,\underline{H}_p(z) = z^{-K}\begin{bmatrix} 0 & 1 \\ z^{-1} & 0 \end{bmatrix}$,

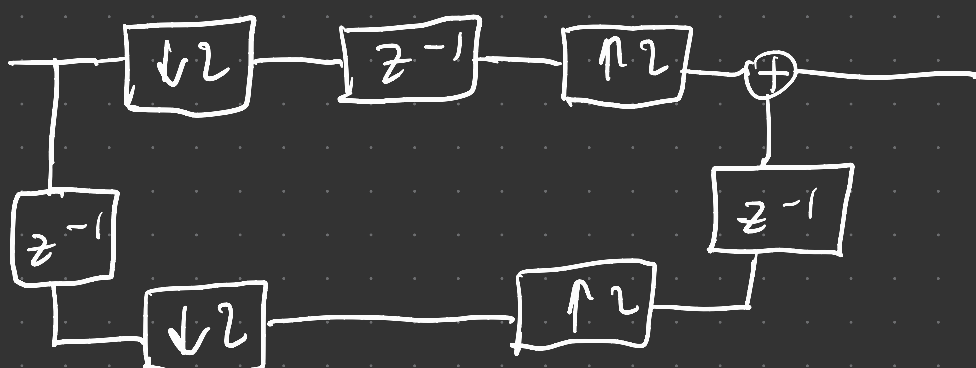
then, PR is guaranteed.

First we will ignore z^{-K}.

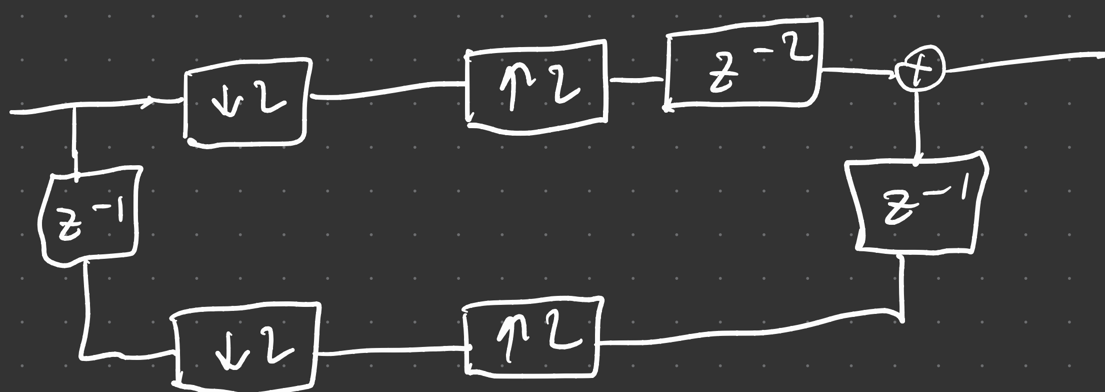
We have the system:



$\equiv$

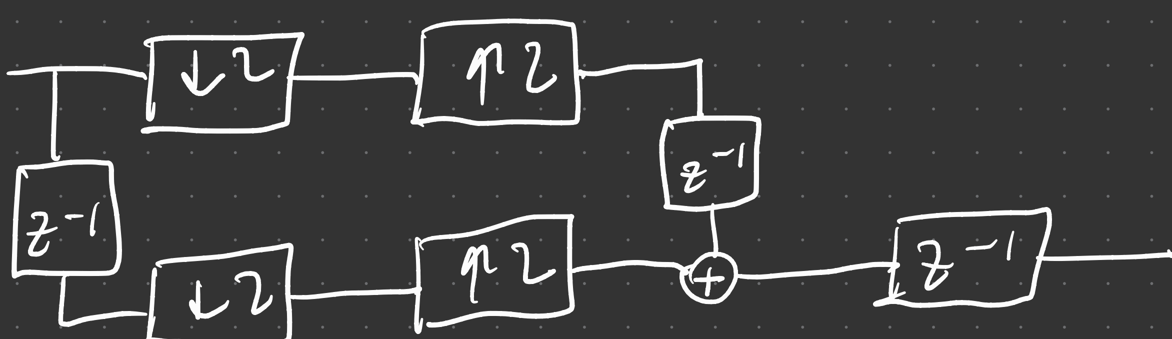


$\equiv$



Noble identity

$\equiv$



$$\equiv \; -\boxed{z^{-1}}-\boxed{z^{-1}}- \quad \equiv \; -\boxed{z^{-2}}\rightarrow$$

Therefore, if $\underline{F}_p(z)\,\underline{H}_p(z) = z^{-k}\begin{bmatrix} 0 & 1 \\ z^{-1} & 0 \end{bmatrix}$,

then $y[n] = x[n-L]$, where $L = 2k+2$.

Orthogonal Filter Banks

Ex: Haar wavelet

$$H_0(z) = F_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$$

$$H_1(z) = \frac{1}{\sqrt{2}}(1-z^{-1})$$

$$F_1(z) = \frac{1}{\sqrt{2}}(-1+z^{-1})$$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

$$= \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{\underline{H}_p(z^2):\ \text{orthogonal matrix}}\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

Obs: • $H_p(z^2)$ is independent of z

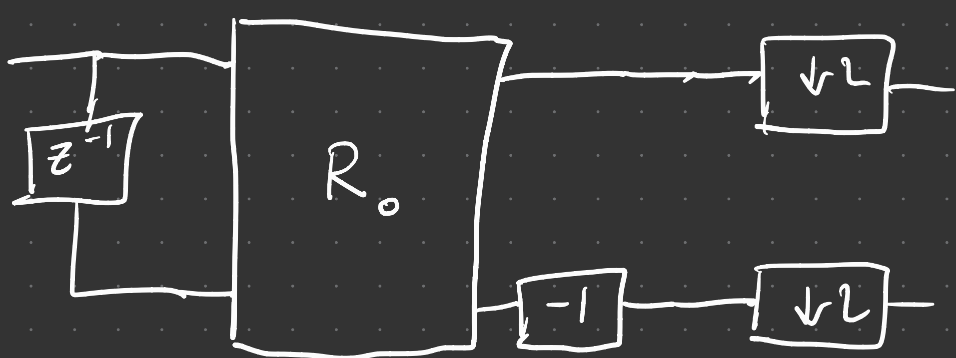
• $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ is a rotation matrix

$$\begin{cases} F_0(z) = z^{-1} H_0(z^{-1}) \\ F_1(z) = z^{-1} H_1(z^{-1}) \end{cases}$$

If H_0 is min-phase, then F_0 is max-phase and vice-versa.

If $H_0(z)$ has all its zeros inside the unit circle, then $H_0(z^{-1})$ will flip them all to be outside, and vice-versa.

In general, for 1st-order systems, the analysis bank takes the form



$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

rotation matrix

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

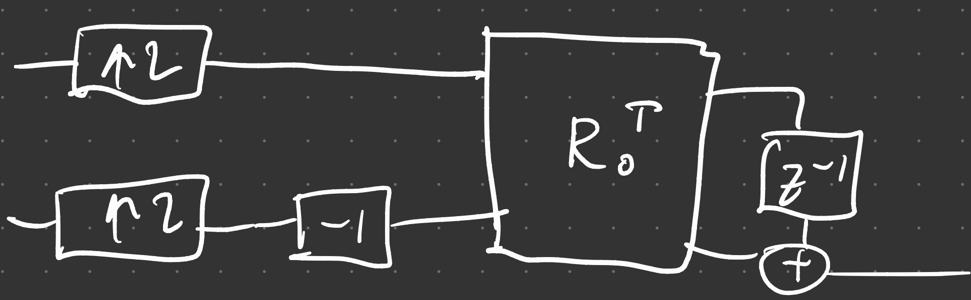
$$= \begin{bmatrix} c_0 & s_0 \\ s_0 & -c_0 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} c_0 + z^{-1} s_0 \\ s_0 - z^{-1} c_0 \end{bmatrix}$$

Obs: In the Haar case, $c_0 = \frac{1}{\sqrt{2}} \Rightarrow \theta_0 = \frac{\pi}{4}$.

Obs: $H_1(z) = -z^{-1} H_0(-z^{-1})$

In general, for 1st order systems, the synthesis bank takes the form



$$[F_0(z) \quad F_1(z)] = [z^{-1} \quad 1] \begin{bmatrix} c_0 & -s_0 \\ s_0 & c_0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= [z^{-1} \quad 1] \begin{bmatrix} c_0 & s_0 \\ s_0 & -c_0 \end{bmatrix}$$

$$= [s_0 + z^{-1} c_0 \qquad -c_0 + z^{-1} s_0]$$

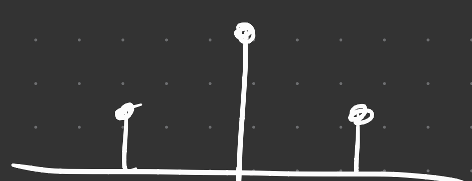
Obs: $\begin{cases} F_0(z) = z^{-1} H_0(z^{-1}) \\ F_1(z) = z^{-1} H_1(z^{-1}) \end{cases}$

Recall: $P_0(z) = F_0(z) H_0(z)$ is a half-band filter.

$$= (s_0 + z^{-1} c_0)(c_0 + z^{-1} s_0)$$

lattice structure

$$= s_0 c_0 + z^{-1} + s_0 c_0 z^{-2}$$



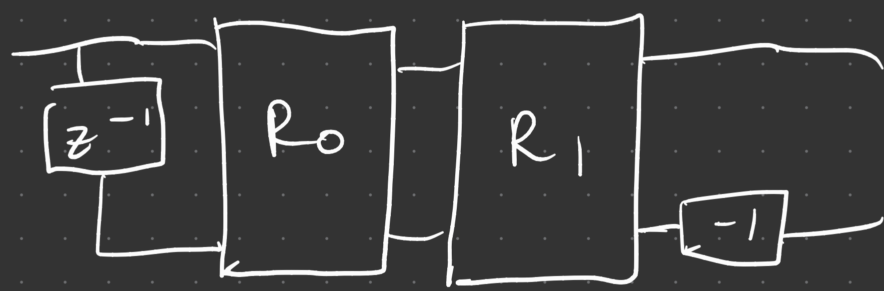
type I linear phase and half band

Remark: This is the general form of 1st order orthogonal filter bank.

Obs: From H_0 you know everything (H_1, F_0, F_1).

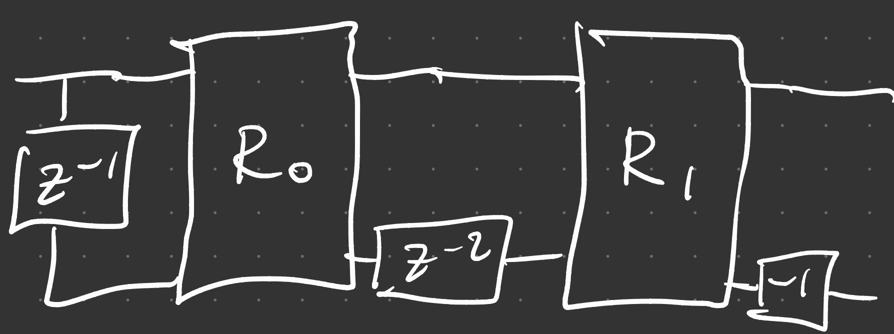
Higher-order Systems

Q: Is this a higher-order filter bank?



A: No. A cascade of two rotations is still a rotation.

Sol^n: Add some delays.



$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-2} \end{bmatrix} \begin{bmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

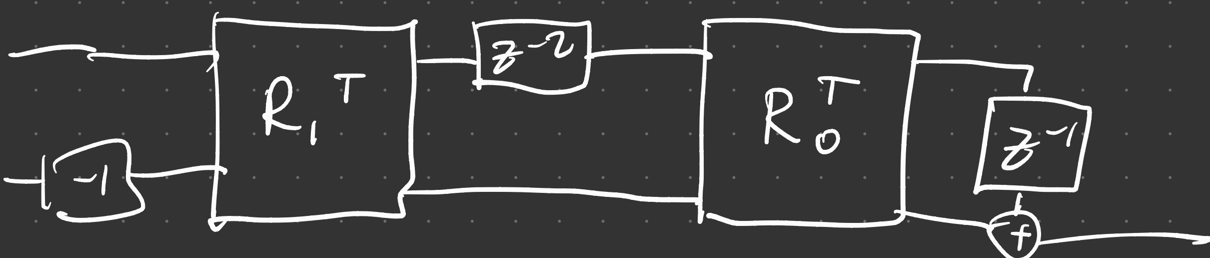
$$= \begin{bmatrix} c_1 & s_1 \\ s_1 & -c_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-2} \end{bmatrix} \begin{bmatrix} c_0 + z^{-1} s_0 \\ -s_0 + z^{-1} c_0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & s_1 \\ s_1 & -c_1 \end{bmatrix} \begin{bmatrix} c_0 + z^{-1} s_0 \\ -z^{-2} s_0 + z^{-3} c_0 \end{bmatrix}$$

$$= \begin{bmatrix} c_0 c_1 + z^{-1} s_0 c_1 - z^{-2} s_0 s_1 + z^{-3} c_0 s_1 \\ c_0 s_1 + z^{-1} s_0 s_1 + z^{-2} s_0 c_1 - z^{-3} c_0 c_1 \end{bmatrix}$$

<u>Obs</u>: $H_1(z) = -z^{-3} H_0(-z^{-1})$

For the synthesis bank, we have



$$\begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \end{bmatrix} \begin{bmatrix} c_0 & -s_0 \\ s_0 & c_0 \end{bmatrix} \begin{bmatrix} z^{-2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} s_0 + c_0 z^{-1} & c_0 - s_0 z^{-1} \end{bmatrix} \begin{bmatrix} z^{-2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & s_1 \\ s_1 & -c_1 \end{bmatrix}$$

$$= \begin{bmatrix} s_0 + c_0 z^{-1} & c_0 - s_0 z^{-1} \end{bmatrix} \begin{bmatrix} z^{-2} c_1 & z^{-2} s_1 \\ s_1 & -c_1 \end{bmatrix}$$

$$= \Big[c_0 s_1 - z^{-1} s_0 s_1 + z^{-2} s_0 c_1 + z^{-3} c_0 c_1$$

$$- c_0 c_1 + z^{-1} s_0 c_1 + z^{-2} s_0 s_1 + z^{-3} c_0 s_1 \Big\}$$

<u>Obs:</u>
$$\begin{cases} F_0(z) = z^{-3} H_0(z^{-1}) \\ F_1(z) = z^{-3} H_1(z^{-1}) \end{cases}$$

<u>Remark</u>: For orthogonal filter banks, you only need to design H_0.

In general, you can get higher-order systems from lower-order systems by cascading more R_K blocks with delays.

(Proof is by induction)

For orthogonal systems:

$$H_p(z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_K \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} R_{K-1} \cdots \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} R_0$$

<u>Exercise:</u> write down $F_p(z)$.

Today: Given θ_l, $l = 1, \dots, K$, find H_0, H_1, F_0, F_1.

Next time: Given H_0, H_1, F_0, F_1 that specify an orthogonal filter bank, find θ_l, $l = 1, \dots, K$.

Exercise: Suppose $H_1(z) = H_0(-z)$, H_0 is FIR. Find all PR systems.