Last Time: Orthogonal Filter Banks

<u>Exercise:</u> Suppose that $H_1(z) = H_0(-z)$, where H_0 is FIR. Find all PR systems.

* Haar wavelet / FB

$$H_0(z) = \frac{1}{\sqrt{2}}(1+z^{-1})$$

$$H_1(z) = \frac{1}{\sqrt{2}}(1-z^{-1})$$

$\longrightarrow$ at least one solution exists

<u>Sol$^{\underline{n}}$:</u> Use the polyphase representation.

$$H_0(z) = \underbrace{E_0(z^2)}_{\text{even polyphase}} + z^{-1} \underbrace{E_1(z^2)}_{\text{odd polyphase}}$$

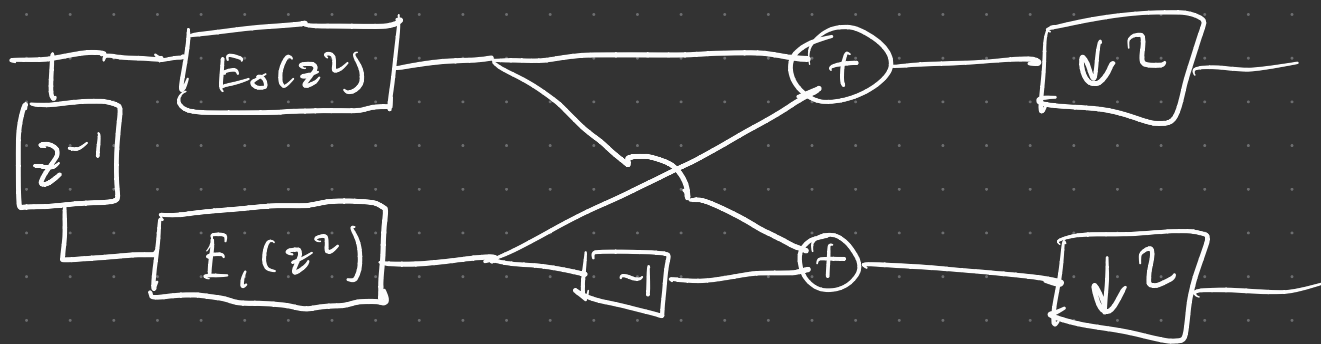
$$H_1(z) = H_0(-z) = E_0(z^2) - z^{-1} E_1(z^2)$$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \underbrace{\begin{bmatrix} E_0(z^2) & E_1(z^2) \\ E_0(z^2) & -E_1(z^2) \end{bmatrix}}_{\underline{H}_p(z^2)} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

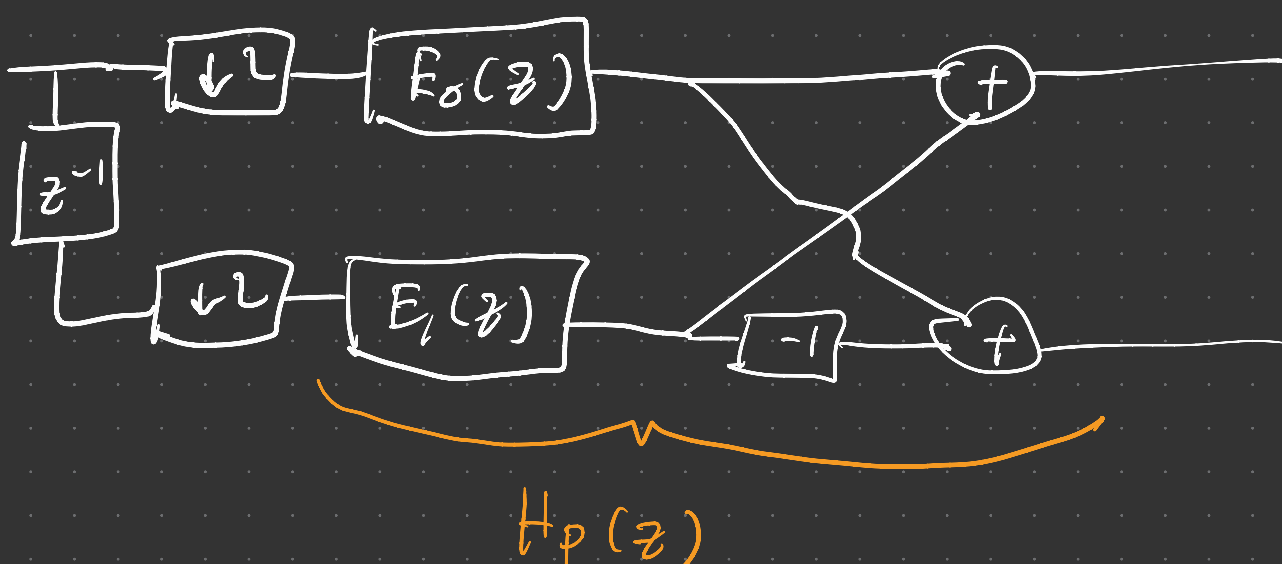
$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0(z^2) & 0 \\ 0 & E_1(z^2) \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

2-point DFT

Analysis Bank:



By the Noble identities

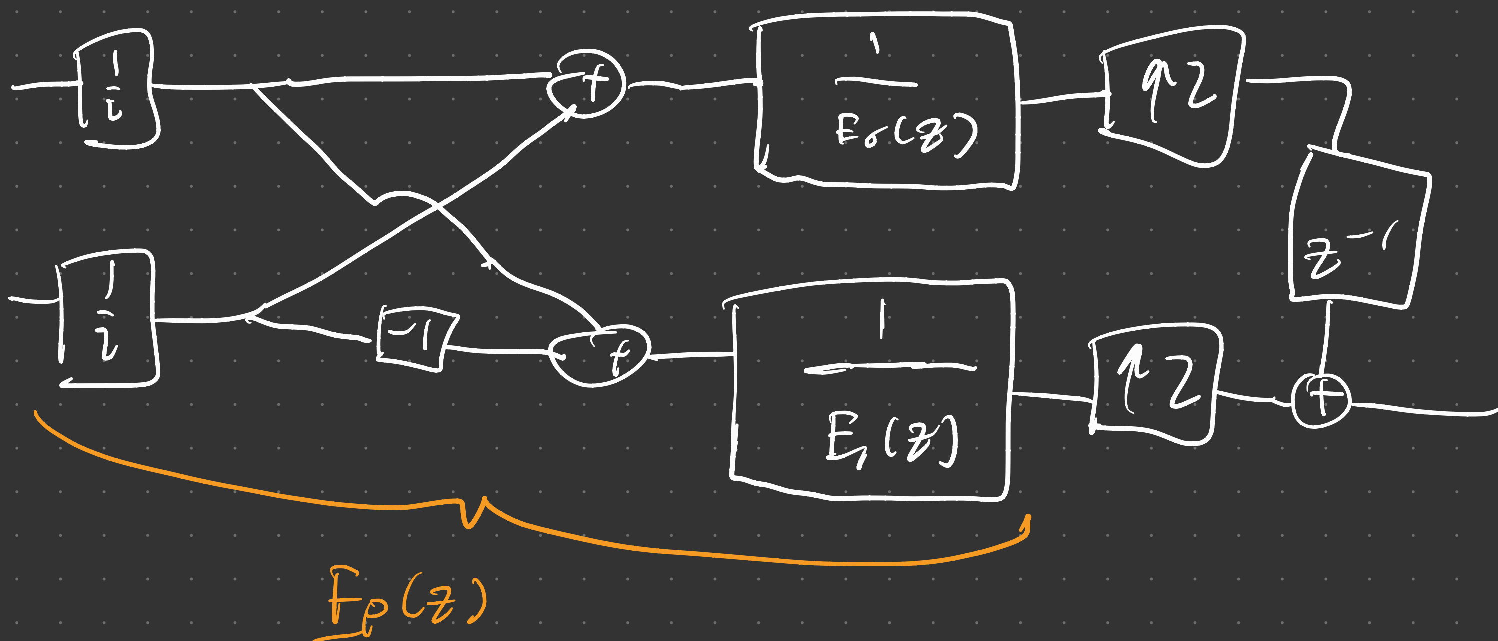


The problem has been reduced to inverting $\underline{H}_p(z)$.
i.e., finding $\underline{F}_p(z)$.

Synthesis Bank:

Observe that

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\underline{F}_p(z) = \frac{1}{2} \begin{bmatrix} \frac{1}{E_0(z)} & 0 \\ 0 & \frac{1}{E_1(z)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} \frac{1}{E_0(z^2)} & 0 \\ 0 & \frac{1}{E_1(z^2)} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} z^{-1} & 1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} \frac{1}{E_0(z^2)} & \frac{1}{E_0(z^2)} \\ \frac{1}{E_1(z^2)} & -\frac{1}{E_1(z^2)} \end{bmatrix}$$

$$= \frac{1}{2} \left[\frac{1}{E_1(z^2)} + z^{-1} \frac{1}{E_0(z^2)} \quad\quad -\frac{1}{E_1(z^2)} + z^{-1} \frac{1}{E_0(z^2)} \right]$$

<u>Remark</u>: So far, we have not assumed <u>anything</u> about the filters.

<u>Q</u>: What if we want FIR filters?

<u>A</u>: Polyphase components must be delays.

i.e.) $E_0(z) = a z^{-l_0}$, $E_1(z) = b z^{-l_1}$

The general FIR solution takes the form:

$$\boxed{\begin{aligned} H_0(z) &= E_0(z^2) + z^{-1} E_1(z^2) \\ &= a z^{-2l_0} + b z^{-(2l_1+1)} \end{aligned}}$$

Obs: For Haar wavelets, $l_0 = 0$, $l_2 = 0$,
$a = \frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$

- For IIR solutions, there are infinite possibilities.

- For causal & stable IIR solution, E_0 & E_1 must be min-phase.

(zeros of E_0 & E_1 are inside unit circle $\Rightarrow$ poles of $\frac{1}{E_0}$ & $\frac{1}{E_1}$ are inside unit circle)

Orthogonal Filter Banks

$$\Lambda(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}$$

Last time we saw that, for an orthogonal FB,

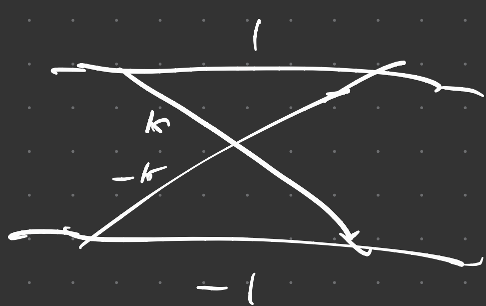
$$\underline{H}_p(z) = \Lambda(-1) R_K \Lambda(z) R_{K-1} \Lambda(z) \cdots \Lambda(z) R_0,$$

where

$$R_\ell = \begin{bmatrix} \cos\theta_\ell & \sin\theta_\ell \\ -\sin\theta_\ell & \cos\theta_\ell \end{bmatrix} \quad \text{is a rot. matrix.}$$

$(K+1)$ parameters vs. $(4K+2)$ for a general factorization.

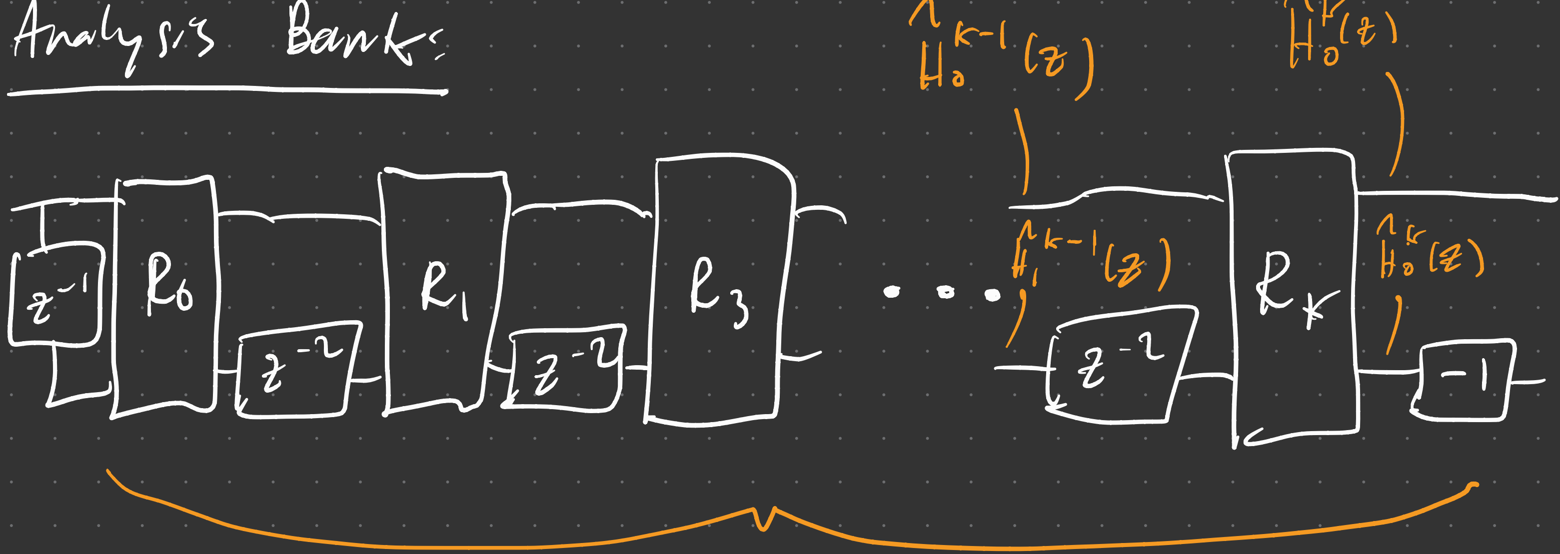
Lattice structure reduces the # of parameters.



$$R_\ell = \cos\theta \begin{bmatrix} 1 & k \\ -k & 1 \end{bmatrix}$$

$$k = \frac{\sin\theta}{\cos\theta}$$

Analysis Bank:



$H_p(z^2)$: polyphase matrix for H_0 & H_1.

order $2K+1$

order $2K-1$

$$\begin{bmatrix} \hat{H}_0^{k}(z) \\ \hat{H}_1^{k}(z) \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \hat{H}_0^{k-1}(z) \\ z^{2}\hat{H}_1^{k-1}(z) \end{bmatrix}$$

$$= \begin{bmatrix} c\,\hat{H}_0^{k-1}(z) + z^{-2} s\,\hat{H}_1^{k-1}(z) \\ -s\,\hat{H}_0^{k-1}(z) + z^{-2} c\,\hat{H}_1^{k-1}(z) \end{bmatrix}$$

Obs: Given $\hat{H}_0^{k-1}(z)$, $\hat{H}_1^{k-1}(z)$, & θ_k, we can find $\hat{H}_0^{k}(z)$ & $\hat{H}_1^{k}(z)$.

lower order $\longrightarrow$ higher order

Q: What about the reverse?

Given $\hat{H}_0^k(z)$ & $\hat{H}_1^k(z)$, can we find

θ_k, $\hat{H}_0^{k-1}(z)$, & $\hat{H}_1^{k-1}(z)$?

Obs: If we can do this, we have shown that the structure above can implement all orthogonal filter banks.

① $\hat{h}_0^k[2k+1] = s\,\hat{h}_1^{k-1}[2k-1]$

impulse response of $\hat{H}_0^k$

② $\hat{h}_0^k[2k] = s\,\hat{h}_1^{k-1}[2k-2]$

highest-order terms

③ $\hat{h}_0^k[0] = c\,\hat{h}_0^{k-1}[0]$

④ $\hat{h}_0^k[1] = c\,\hat{h}_0^{k-1}[1]$

lowest-order terms

<u>Recall:</u> For orthogonal filter banks

$$\hat{H}_1^{k-1}(z) = z^{-(2k-1)} \hat{H}_0^{k-1}(-z^{-1}) \quad 5$$

- $\hat{h}_1^{k-1}[2k-1] = -\hat{h}_0^{k-1}[0]$ —— 5

$$= \frac{\hat{h}_0^{k}[2k+1]}{s} \quad \text{—— } 1$$

- $\hat{h}_0^{k-1}[0] = \dfrac{\hat{h}^{k}[0]}{c}$ —— 3

$$\Rightarrow -\frac{\hat{h}_0^{k}[2k+1]}{s} = \frac{\hat{h}_0^{k}[0]}{c}$$

$$\Rightarrow \underbrace{\frac{s}{c}}_{\tan\theta_k} = -\frac{\hat{h}_0^{k}[2k+1]}{\hat{h}_0^{k}[0]}$$

$$\boxed{\theta_k = \tan^{-1}\left(-\frac{\hat{h}_0^{k}[2k+1]}{\hat{h}_0^{k}[0]}\right)}$$

- $\hat{h}_1^{k-1}[2k-2] = \hat{h}_0^{k-1}[1]$ ⟶ S

$$= \frac{\hat{h}_0^{k}[2k]}{S} \longrightarrow 2$$

- $\hat{h}_0^{k-1}[1] = \dfrac{h_0^k[1]}{C}$ ⟶ 4

$$\Rightarrow \frac{\hat{h}_0^{k}[2k]}{S} = \frac{\hat{h}_0^{k}[1]}{C}$$

$$\tan\theta_k = \frac{S}{C} = \frac{\hat{h}_0^{k}[2k]}{\hat{h}_0^{k}[1]}$$

$$\boxed{\theta_k = \tan^{-1}\left(\frac{\hat{h}_0^{k}[2k]}{\hat{h}_0^{k}[1]}\right)}$$

Q: Is

$$-\frac{\hat{h}_0^k[2k+1]}{\hat{h}_0^k[0]} = \frac{\hat{h}_0^k[2k]}{\hat{h}_0^k[1]} \ ?$$

Alternatively, is

$$☆\ \hat{h}_0^k[1]\,\hat{h}_0^k[2k+1] + \hat{h}_0^k[2k]\,\hat{h}_0[0] = 0 \ ?$$

for orthogonal systems?

Hint: $P_0(z) = F_0(z)\,H_0(z)$

Since the system is orthogonal,

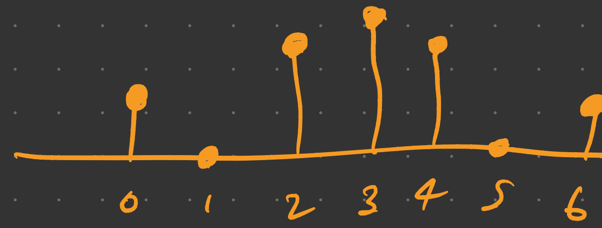
$$F_0(z) = z^{-(2k+1)} H_0(z^{-1})$$

$\Rightarrow$ $P_0(z) = z^{-(2k+1)} H_0(z^{-1})\,H_0(z)$ ☆

is half-band.

Claim: ☆ implies ☆

Note: $h_0^K = h_0$.



$H_0(z)$: $h_0[0]$ $h_0[1]$ $h_0[2]$ $\cdots$ $h_0[2k]$ $h_0[2k+1]$

$z^{-(2k+1)} H_0(z^{-1})$: $h_0[2k+1]$ $h_0[2k]$ $h_0[2k-1]$ $\cdots$ $h_0[1]$ $h_0[0]$

$$z^{-1} : h_0[0]\, h_0[2k] + h_0[1]\, h_0[2k+1] = 0 \checkmark$$

half band

Obs: This only works because the system is orthogonal & P_0 is half-band.

At this point we have θ_K, which allows us to construct $\hat{H}_0^{k-1}$ & $\hat{H}_1^{k-1}$.

Exercise: $H_1(z) = H_0(-z)$.

If $H_0(z)$ is Type I linear-phase, are F_0 & F_1 causal & stable? No.

Sol$^{\underline{n}}$: Causal & stable F_0 & F_1

$\Rightarrow$ E_0 & E_1 are min-phase.

Ex: $h_0[n]$: a b c d c b a

$e_0[n]$: a c c a

$e_1[n]$: b d b

polyphase components of Type I linear-phase system are linear-phase.

linear-phase systems cannot be min-phase.

Q: What if H_0 is Type II linear-phase?

No.

Ex: $h_0[n]$: a b c d | d c b a

$e_0[n]$: a c d b

$e_1[n]$: b d c a

polyphase components of Type II linear-phase system are flips of each other.

$$E_1(z) = z^{-L} E_0(z^{-1})$$

If one of them is min-phase, the other is max-phase.