Today: Multiresolution, Wavelets, Filter Banks

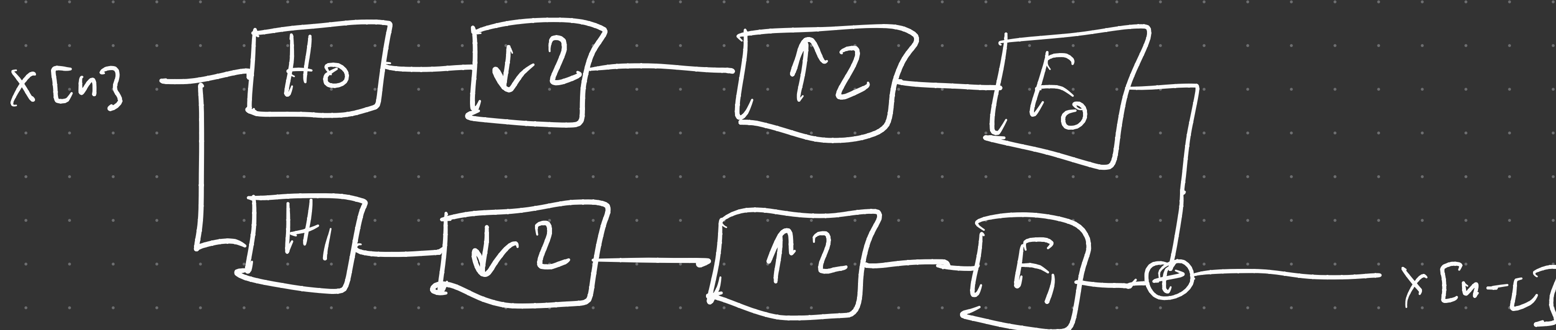
Multiresolution: unifying concept

Wavelets: continuous-time

Filter Banks: discrete-time

Discrete Wavelet Transform (DWT)

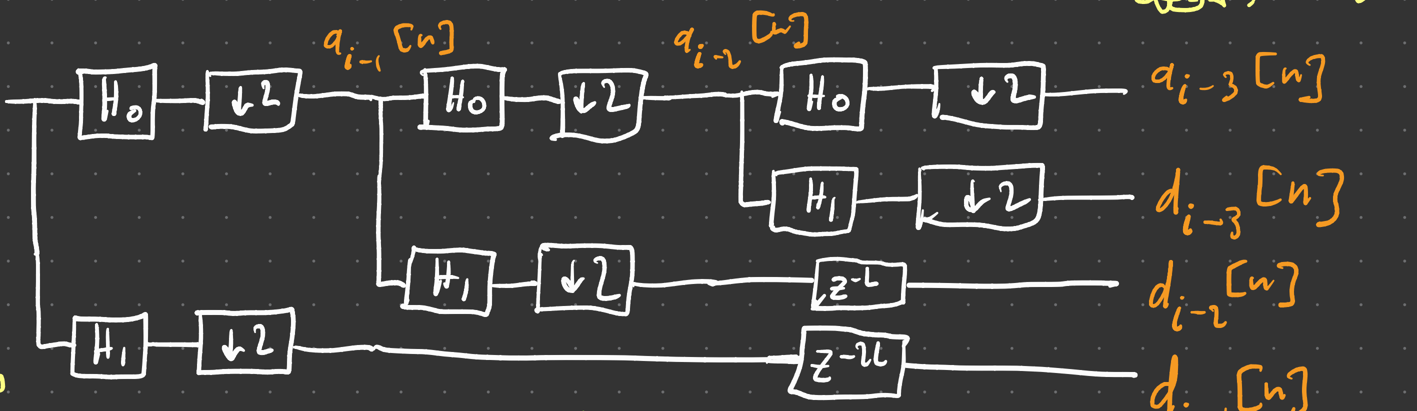
- (Dyadic) DWTs are based on two-channel PR filter banks.



- DWTs are based on iterated structures.

approx / scaling coeff.

discrete approx of an analogue object



Wavelet Analysis Bank

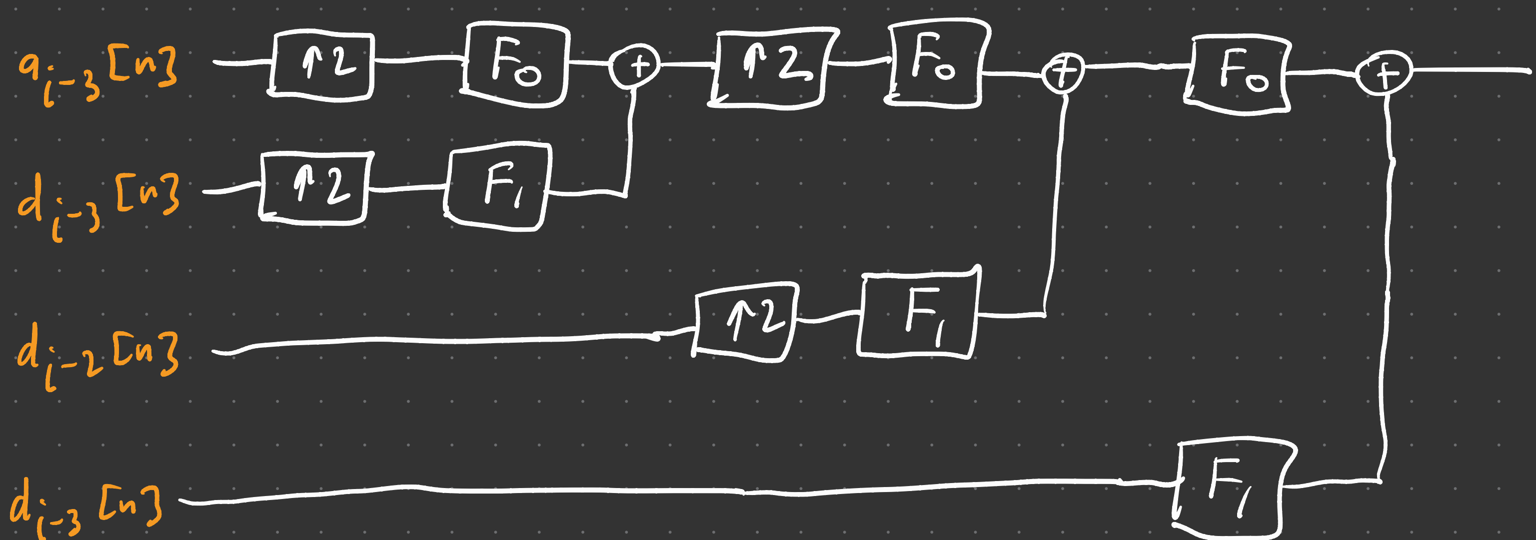
(Dyadic) DWT

detail / wavelet coeff.

Q: Given the approximation and detail coefficients, can we recover the original approx. of the analog signal?

Concretely: given $d_{i-1}[n]$, $d_{i-2}[n]$, $d_{i-3}[n]$, and $a_{i-3}[n]$, can we get back $a_i[n]$?

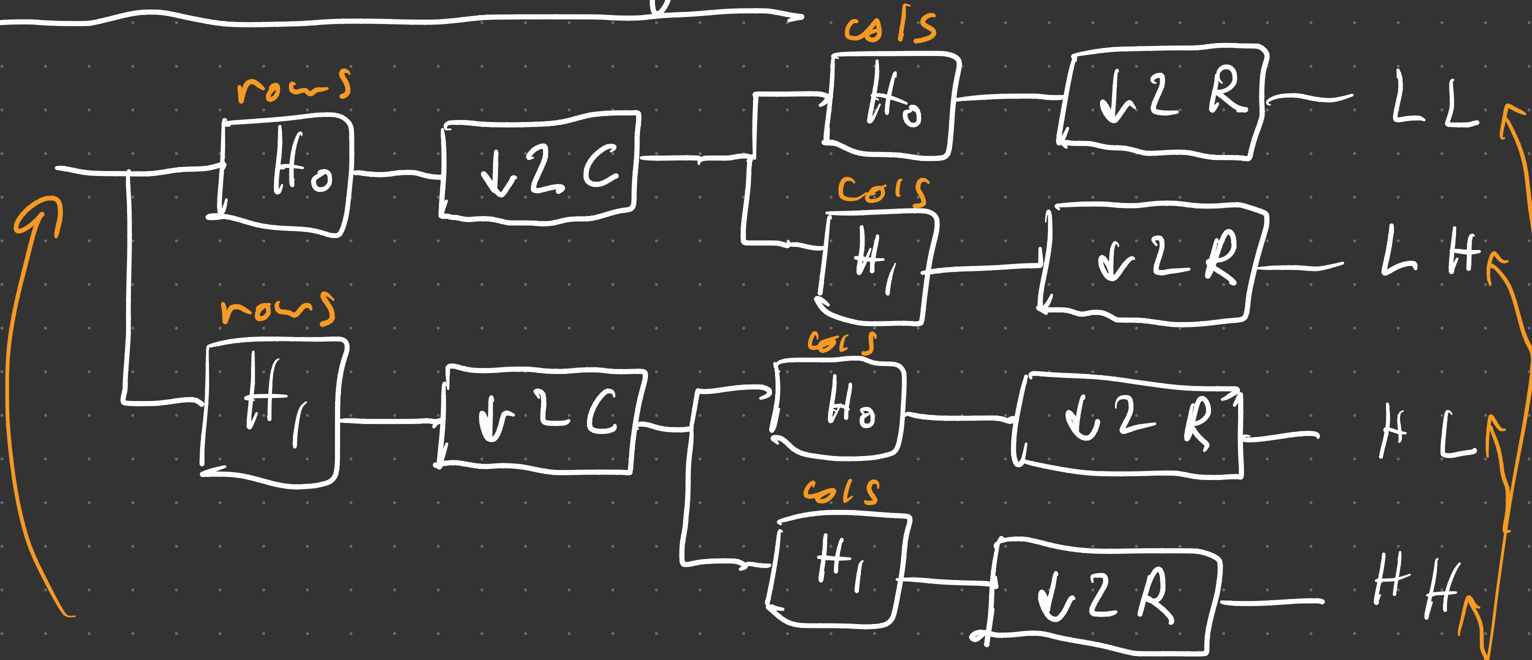
A: Yes.



Wavelet Synthesis Bank
Inverse DWT

Obs: The DWT is a multiresolution decomp. of the input signal.

DWT in 2D (Images)

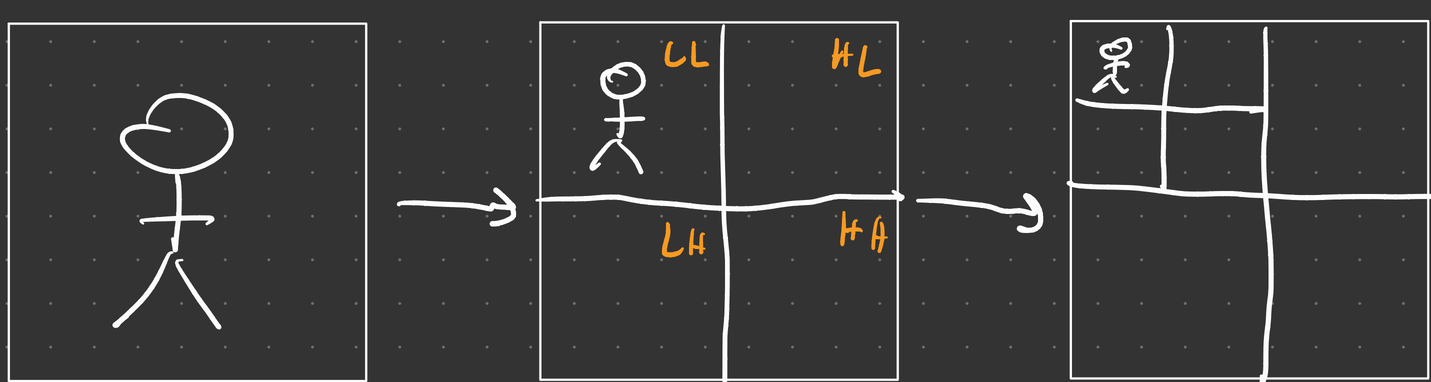


$(2N)\times(2N)$ image

$N\times N$

Obs: Total # of coeffs are the same at the front and back.

The 2D DWT iterates this procedure on LL.



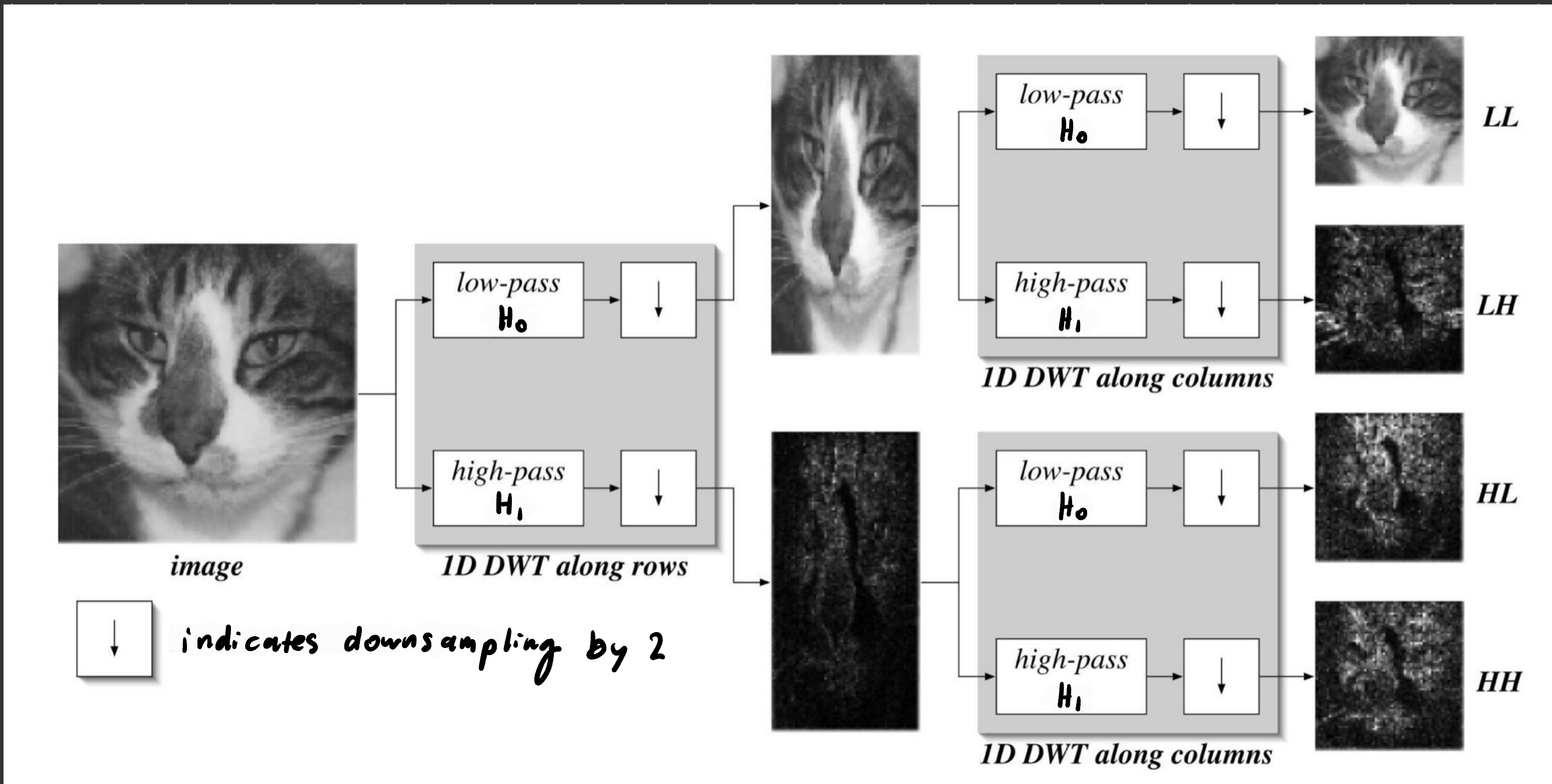
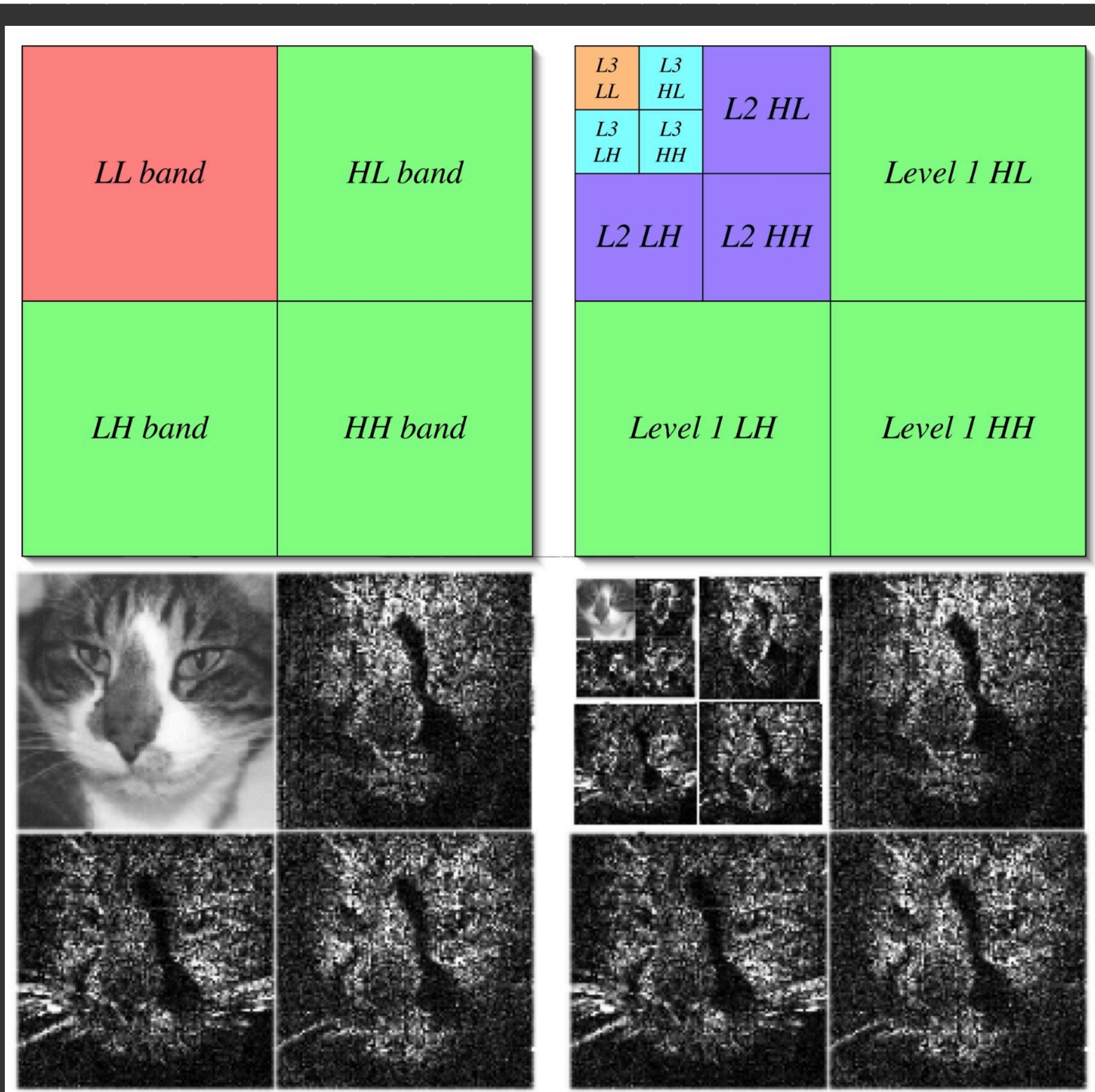


image
low-pass
H0
high-pass
H1
1D DWT along rows
low-pass
H0
high-pass
H1
1D DWT along columns
low-pass
H0
high-pass
H1
1D DWT along columns
LL
LH
HL
HH
indicates downsampling by 2


LL band
HL band
LH band
HH band
L3 LL
L3 HL
L3 LH
L3 HH
L2 HL
L2 LH
L2 HH
Level 1 HL
Level 1 LH
Level 1 HH

Remark: For real-world signals, many of the scaling/wavelet coefficients are small or zero, even with noise.

"sparsity"

→ Easy denoising, compression, etc. via simple <u>thresholding procedures</u>.

→ Thresholding is <u>non linear</u>.

(r)evolution of sparsity

- wavelet shrinkage/thresholding algorithms (Donoho et al., 1990s)
- compressed sensing (Candes, Romberg, Tao, Donoho, 2006)
- Deep learning (2010 – now)

Q: Where does the discrete approx. of our analog signal come from?

A: Acquisition or sampling device.

(Generalized) Sampling of Analog Signals

$f(t)$, $t \in \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$

Ideal sampling:

$$a[n] = f(nT) = \int_{-\infty}^{\infty} f(t)\, \underbrace{\delta(t-nT)}_{\delta_{nT}(t)}\, dt$$

approximation

$$= \langle f, \delta_{nT} \rangle$$

inner product

Whittaker–Nyquist–Kotelnikov–Shannon Sampling Theorem: If $f(t)$ is bandlimited to $[-B, B]$, then

$$f(t) = \sum_{n \in \mathbb{Z}} x[n] \frac{\sin(B(t-nT))}{B(t-nT)},$$

so long as $T < \frac{\pi}{B}$.

Def$^{\underline{n}}$: $f(t)$ is band limited to $[-B, B]$ if its Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

angular frequency

has support in $[-B, B]$, i.e.,

"closure"

$$\overline{\operatorname{supp} F = \{\omega \in \mathbb{R} : F(\omega) > 0\}}$$

satisfies

$$\operatorname{supp} F \subseteq [-B, B].$$

Remark: We are not indexing freq. in Hz.

If we write $\omega = 2\pi f$ — Hz, we can define $\tilde{B} = \frac{B}{2\pi}$ in Hz and the condition becomes $T < \frac{1}{2\tilde{B}}$ or that the sampling rate satisfies $\tilde{B} < \frac{f_s}{2}$, where $f_s = \frac{1}{T}$.

Q: How do you invert the Fourier transform?

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega$$

Generalized (Non-Ideal) Sampling:

$$a[n] = \langle f, \varphi_n \rangle = \int_{-\infty}^{\infty} f(t) \underbrace{\varphi_n(t)}_{\varphi(t-n) \ [\text{i.e., } T=1]} \, dt$$

approximation

- φ is modeling the impulse response of the acquisition device

→ if $\tilde{\varphi}(t) = \varphi(-t)$

$$\langle f, \varphi_n \rangle = (f * \tilde{\varphi})(n)$$

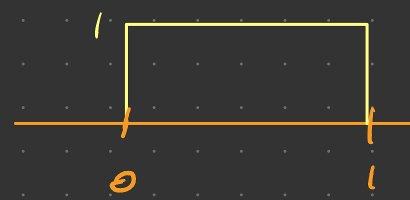
$$= \int_{-\infty}^{\infty} f(t) \tilde{\varphi}(n-t) \, dt$$

$$= \int_{-\infty}^{\infty} f(t) \varphi(t-n) \, dt \ ✓$$

Q: What's the simplest choice of φ?

A: Box (or rect) function:

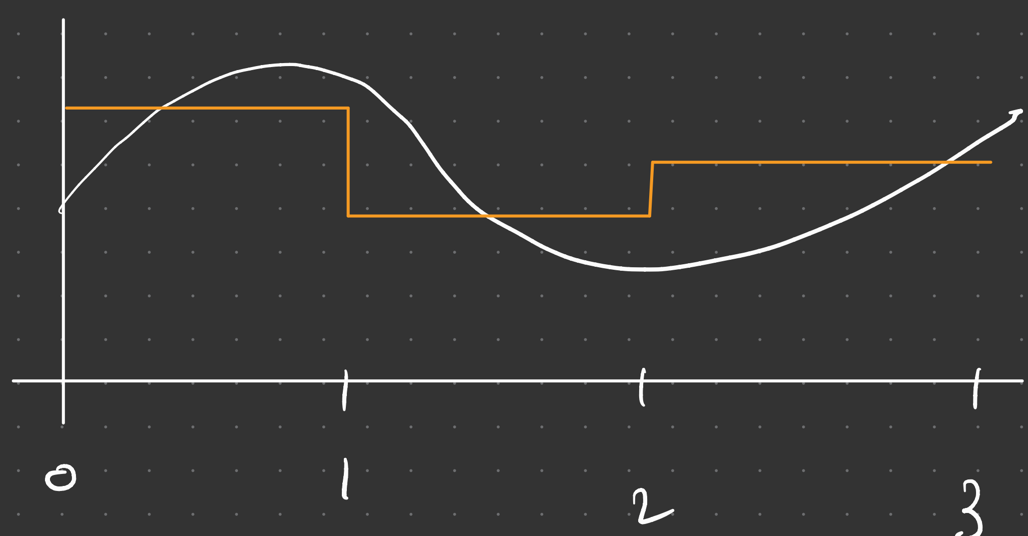
$$\varphi(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{else} \end{cases}$$



Q: What happens if we sample an analog signal with $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$?

$i=0$

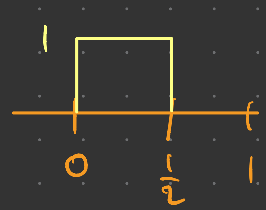
A: Piecewise constant approx.



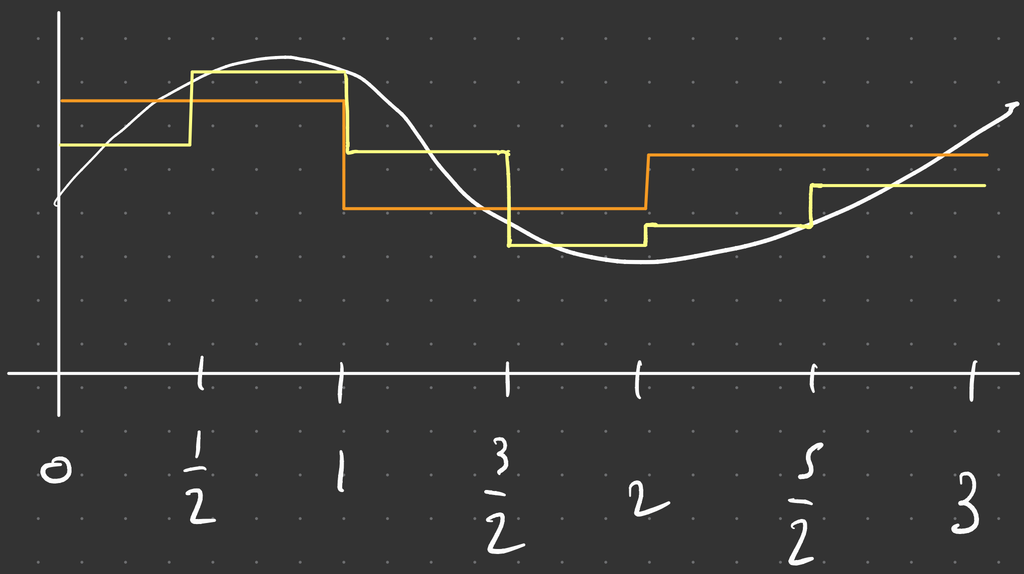
Q: What if we want a higher resolution?
How do we double the resolution?

$i=1$

A: Sample with $\{\varphi(2t-n)\}_{n\in\mathbb{Z}}$

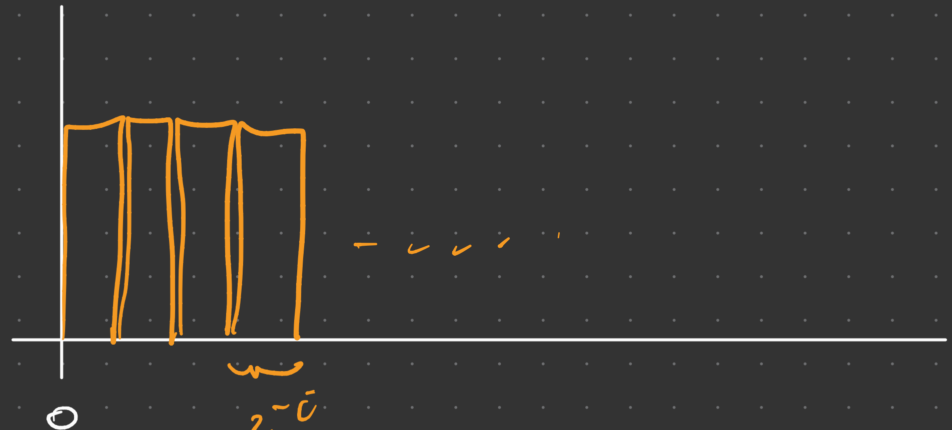


We get a higher-resolution approximation



Obs: If we keep doubling the resolution, we can recover the original analog signal!!!

Remark: For a general resolution i, we sample with $\{\varphi(2^i t - n)\}_{n\in\mathbb{Z}}$



Q: Is there a problem?

What if $i \to \infty$?

$$\varphi(2^i t - n) \to 0 \text{ for almost every } t$$

Fix: Normalize the sampling functions to have unit energy:

$$\left\{ 2^{\frac{i}{2}} \varphi(2^i t - n) \right\}_{n \in \mathbb{Z},\, i \in \mathbb{Z}}$$

$$\int_{-\infty}^{\infty} \left[2^{\frac{i}{2}} \varphi(2^i t - n) \right]^2 dt$$

$$= 2^i \int_{-\infty}^{\infty} \varphi(2^i t - n)^2 \, dt \qquad \begin{aligned} u &= 2^i t - n \\ du &= 2^i dt \end{aligned}$$

$$= \underbrace{\int_{-\infty}^{\infty} \varphi(u)^2 \, dt}_{\|\varphi\|_{L^2}^2 = \langle \varphi, \varphi \rangle} = 1$$

"Lebesgue space"

Def$^{\underline{n}}$: The space of finite-energy signals is

$$L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} : \|f\|_{L^2} < \infty \}$$