Last Time: Multiresolution and Wavelets

- PR FBs. vs. DWTs.

PR FB:

DWT:

- Iterated structure
- Multiscale decomposition of the input signal.

Remark: All of this processing is in the discrete domain.

$\longrightarrow$ There is an underlying sampling procedure.

Generalized Sampling

Given an analog signal $f : \mathbb{R} \to \mathbb{R}$, we can construct a discrete approximation

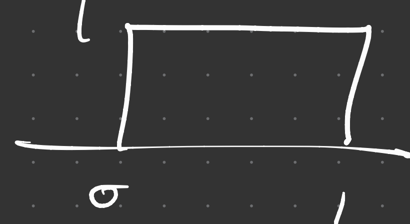
$$a[n] = \langle f, \varphi_n \rangle = \int_{-\infty}^{\infty} f(t)\, \underbrace{\varphi_n(t)}_{\varphi(t-n)}\, dt$$

$n \in \mathbb{Z}$

- φ models the impulse response of the acquisition system.

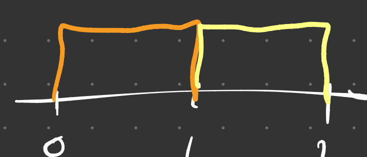
Ex: Box (rect) function

$$\varphi(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & \text{else.} \end{cases}$$

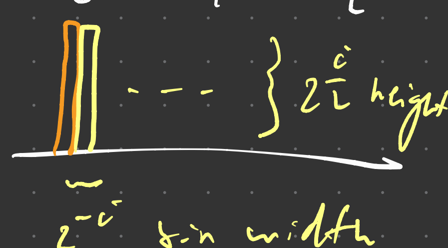


Sampling at different resolutions:

$\{\varphi(t-n)\}_{n \in \mathbb{Z}}$ resolution 0



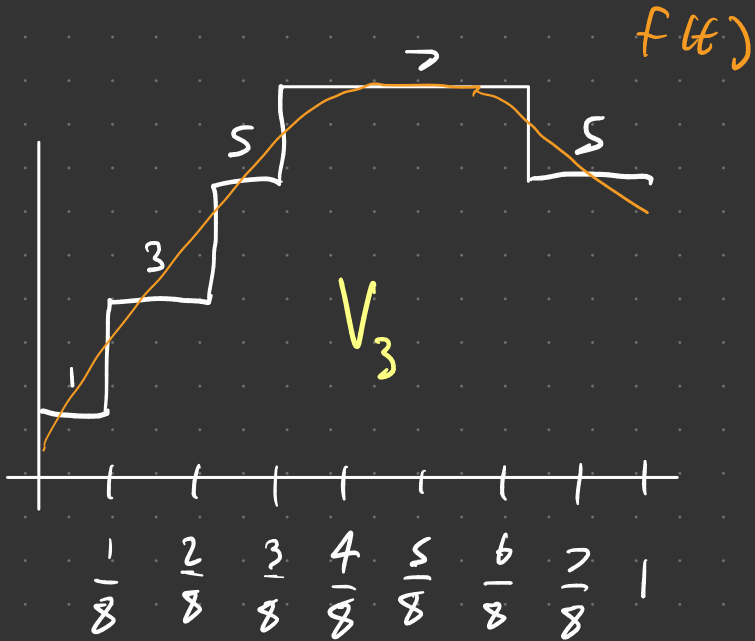
$\{2^{\frac{i}{2}} \varphi(2^i t - n)\}_{n \in \mathbb{Z}}$ resolution i



Ex: Suppose we have a resolution 3 approx. of an analog signal.

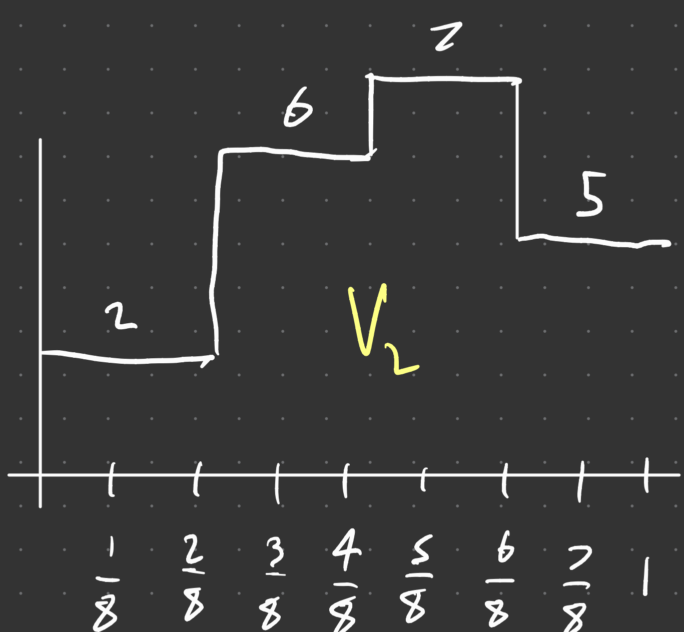
Q: What is the bin width?

A: $2^{-3} = \frac{1}{8}$



$i = 3$ approximation

Basis: $\left\{2^{\frac{3}{2}}\varphi(2^3 t - n)\right\}_{n\in\mathbb{Z}}$



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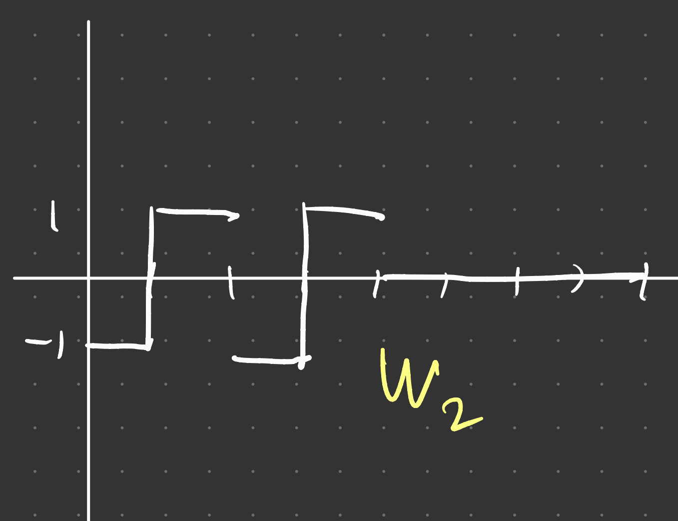
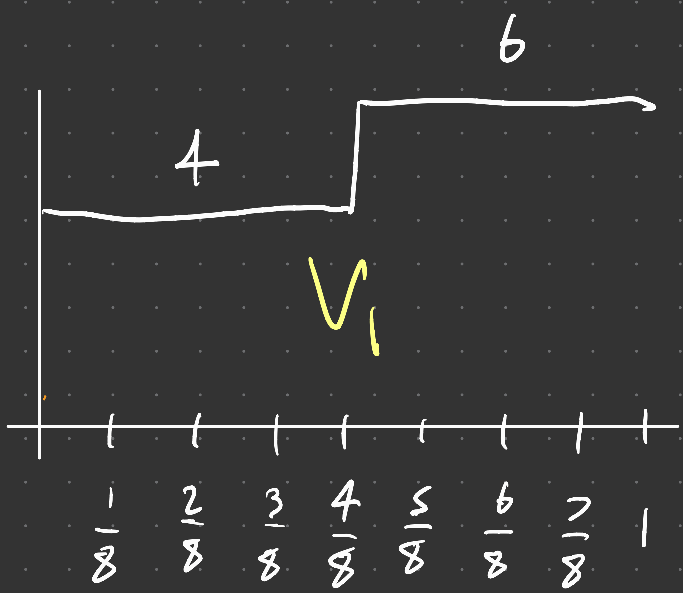


$i = 2$ approximation

Basis: $\left\{2\varphi(2^2 t - n)\right\}_{n\in\mathbb{Z}}$

$\boxed{i = 2 \text{ detail}}$

Basis: $\left\{2\psi(2^2 t - n)\right\}_{n\in\mathbb{Z}}$



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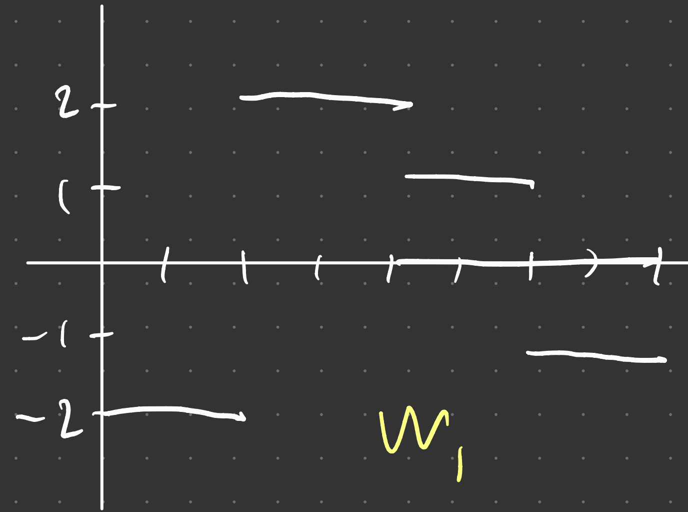
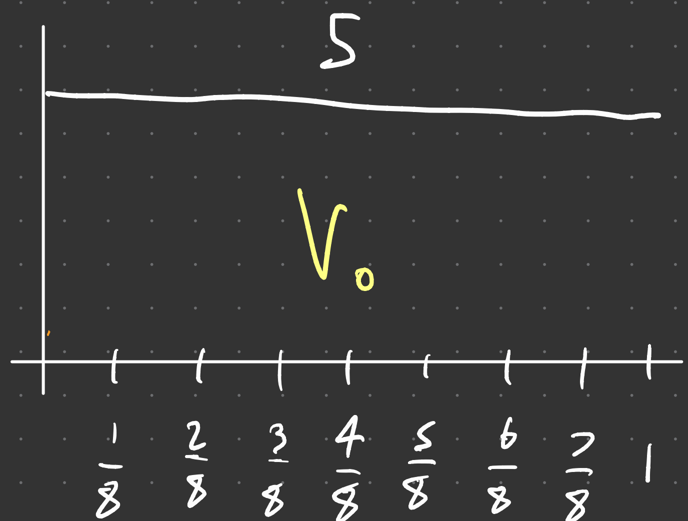


$i = 1$ approximation

Basis: $\{2^{1/2}\varphi(2t-n)\}_{n\in\mathbb{Z}}$

$\boxed{i = 1 \text{ detail}}$

Basis: $\{2^{1/2}\psi(2t-n)\}_{n\in\mathbb{Z}}$



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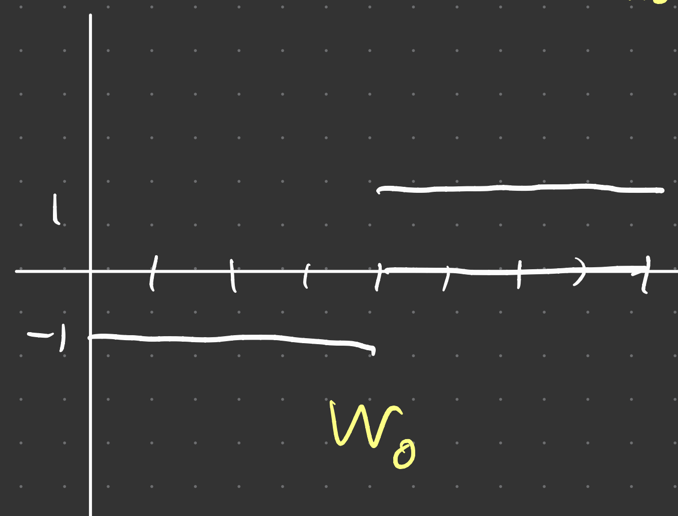


$\boxed{i = 0 \text{ approximation}}$

Basis: $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$

$\boxed{i = 0 \text{ detail}}$

Basis: $\{\psi(t-n)\}_{n\in\mathbb{Z}}$

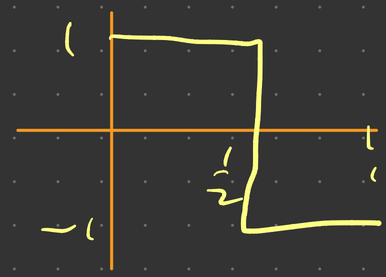
Obs: This is a multiresolution decomp.

Obs: The approximation bases are multiscale versions of $\varphi(t)$.

Q: What basis functions are we using for the details?

A: Multiscale versions of

$$\psi(t) = \begin{cases} +1, & 0 \le t \le \frac{1}{2} \\ -1, & \frac{1}{2} < t \le 1 \\ 0, & \text{else} \end{cases}$$



Remark: $\psi(t)$ is the Haar (mother) wavelet

$\varphi(t)$ is the Haar scaling function (father wavelet).

Defⁿ: The approximation space at resolution i is the space

$$V_i = \text{span}\left\{ 2^{\frac{i}{2}} \varphi(2^i t - n) \right\}_{n \in \mathbb{Z}}$$

Defⁿ: The wavelet (detail) space at resolution i is the space

$$W_i = \text{span}\left\{ 2^{\frac{i}{2}} \psi(2^i t - n) \right\}_{n \in \mathbb{Z}}$$

Q: Which is bigger: V_0 or V_1?

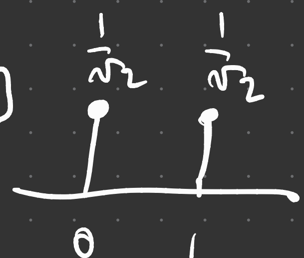
A: $V_0 \subset V_1$

Q: Which is bigger: W_0 or V_1?

A: $W_0 \subset V_1$

What does this mean?

- $\underbrace{\varphi(t)}_{V_0} = \sum_{n \in \mathbb{Z}} \tilde{h}_0[n] \underbrace{\sqrt{2}\,\varphi(2t-n)}_{V_1}$

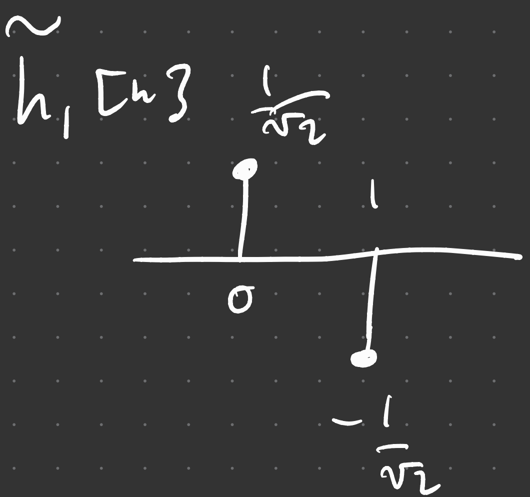
$\tilde{h}_0[n]$ 

Haar FB

low-pass

filter

- $\psi(t) = \sum_{n \in \mathbb{Z}} \tilde{h}_1[n] \sqrt{2}\, \varphi(2t-n)$



Haar FB high-pass filter

Obs: $\tilde{h}_0[n]$ & $\tilde{h}_1[n]$ are the Haar wavelet analysis filters!

Remark: These equations are called the two-scale equations

Q: How do we get V_1 from V_0 & W_0?

A: $V_1 = V_0 \oplus W_0$.

direct sum

- $V_1 = \{ f + g_0 : f \in V_0 \text{ and } g_0 \in W_0 \}$ "sum"

- $V_0 \cap W_0 = \{0\}$ "direct"

Q: What happens to V_i as i becomes large?

A: $V_i \longrightarrow L^2(\mathbb{R})$ as $i \to \infty$

Q: What happens to V_i as i becomes small?

A: $V_i \longrightarrow \{0\}$ as $i \to -\infty$

Obs: We have constructed a nested sequence of subspaces of $L^2(\mathbb{R})$:

$$\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$$

Remark: This sequence is called a multiresolution analysis (MRA) of $L^2(\mathbb{R})$.

MRA (Mallat, Meyer, ca. 1988)

A nested sequence of subspaces $\{V_i\}_{i \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ is called a MRA if:

1. $\overline{\bigcup_{i \in \mathbb{Z}} V_i} = L^2(\mathbb{R})$

2. $\bigcap_{i \in \mathbb{Z}} V_i = \{0\}$

3. $f(t) \in V_0$ if and only if $f(2^i t) \in V_i$

4. $f(t) \in V_0$ implies $f(t-n) \in V_0 \quad \forall n \in \mathbb{Z}$

5. There exists a scaling function $\varphi(t) \in V_0$ such that $\{\varphi(t-n)\}_{n \in \mathbb{Z}}$ is an $\underbrace{\text{orthonormal basis}}_{\text{can be relaxed}}$ for V_0.

Can be relaxed

Riesz basis, frames, etc.

Haar system

Q: IS $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ an ortho basis? YES

For $n, m \in \mathbb{Z}$

$$\langle \varphi(t-n), \varphi(t-m)\rangle = \int_{-\infty}^{\infty} \varphi(t-n)\,\varphi(t-m)\,dt$$

$$= \delta[n-m] \checkmark$$

Exercise: Check the other properties.

Recall: $V_1 = V_0 \oplus W_0$

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$$

$$\vdots$$

$$V_i = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{i-1}$$

Notation: $\varphi_{i,n}(t) = 2^{\frac{i}{2}}\varphi(2^i t - n)$

$$\psi_{i,n}(t) = 2^{\frac{i}{2}}\psi(2^i t - n)$$

Obs: The functions $\{\varphi_{0,n}\}_{n\in\mathbb{Z}} \cup \{\psi_{i,n}\}_{i\geq 0,\, n\in\mathbb{Z}}$ form an ortho basis for $L^2(\mathbb{R})$.

Exercise: Show that

$$\langle \psi_{i,n}, \psi_{i',n'} \rangle = \delta[i-i']\,\delta[n-n']$$

$$\langle \psi_{i,n}, \varphi_{0,m} \rangle = 0$$

Given $f \in L^2(\mathbb{R})$, we can write

$$f(t) = \underbrace{\sum_{n\in\mathbb{Z}} \langle f, \varphi_{0,n} \rangle \varphi_{0,n}(t)}_{\substack{\text{coarse approximation} \\ \text{projection of } f \text{ onto } V_0}} + \underbrace{\sum_{i=0}^{\infty} \sum_{n\in\mathbb{Z}} \langle f, \psi_{i,n} \rangle \psi_{i,n}(t)}_{\substack{\text{details over all} \\ \text{resolutions}}}$$

Synthesis procedure

$$= \sum_{n\in\mathbb{Z}} a_0[n]\, \varphi_{0,n}(t) + \sum_{i=0}^{\infty} \sum_{n\in\mathbb{Z}} d_i[n]\, \psi_{i,n}(t)$$