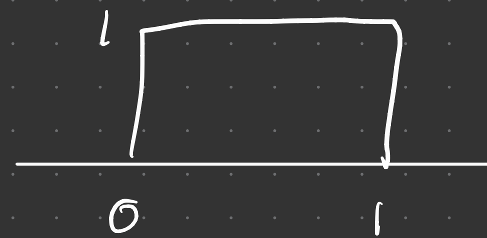
Last Time: MRA of $L^2(\mathbb{R})$ and Haar Wavelets

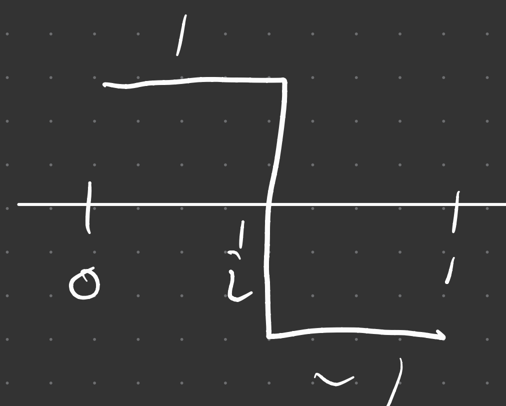
Haar scaling function:

$$\varphi(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{else} \end{cases}$$



Haar wavelet function:

$$\psi(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{2} \\ -1, & \frac{1}{2} < t \le 1 \\ 0, & \text{else} \end{cases}$$



Notation:

$$\varphi_{i,n}(t) = 2^{\frac{i}{2}} \varphi(2^i t - n)$$

$$\psi_{i,n}(t) = 2^{\frac{i}{2}} \psi(2^i t - n)$$

translates and dilates of the scaling and wavelet functions

Approximation and Wavelet Spaces:

$$V_i = \operatorname{Span}\{\varphi_{i,n}\}_{n \in \mathbb{Z}}$$

$$W_i = \operatorname{Span}\{\psi_{i,n}\}_{n \in \mathbb{Z}}$$

The collection $\{V_i\}_{i \in \mathbb{Z}}$ form a nested sequence of subspaces of $L^2(\mathbb{R})$

$$\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$$

called a <u>multiresolution analysis (MRA)</u>.

<u>Recall:</u> $V_0 \subset V_1$, $W_0 \subset V_1$

$$V_1 = V_0 \oplus W_0$$

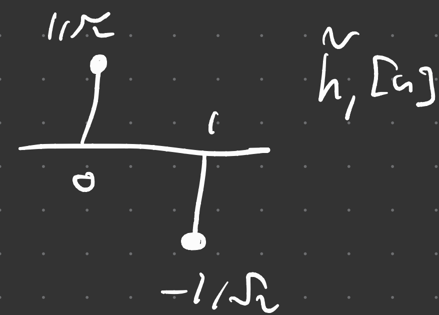
$\longrightarrow$ W_0 is the <u>orthogonal complement</u> of V_0 in V_1.

$$\longrightarrow W_0 = \{f \in V_1 : \langle f, g \rangle = 0, \forall g \in V_0\}$$

<u>Two-Scale Equations:</u>

$$\varphi(t) = \sum_{n \in \mathbb{Z}} \tilde{h}_0[n] \sqrt{2}\, \varphi(2t-n)$$

$$\psi(t) = \sum_{n \in \mathbb{Z}} \tilde{h}_1[n] \sqrt{2}\, \varphi(2t-n)$$



<u>Obs:</u> $\tilde{h}_0[n]$ & $\tilde{h}_1[n]$ are the Haar FB analysis filters.

Last time, we did this by inspection.

<u>Q:</u> Is there a more direct way to do this?
(i.e., for other wavelets?)

<u>A:</u> Projections.

$$\varphi \in V_0 \subset V_1 \Rightarrow \varphi(t) = \mathrm{Proj}_{V_1} \varphi(t)$$

$$= \sum_{n \in \mathbb{Z}} \underbrace{\langle \varphi, \varphi_{1,n} \rangle}_{\tilde{h}_0[n]} \varphi_{1,n}(t)$$

$$\psi \in W_0 \subset V_1 \Rightarrow \psi(t) = \mathrm{Proj}_{V_1} \psi(t)$$

$$= \sum_{n \in \mathbb{Z}} \underbrace{\langle \psi, \varphi_{1,n} \rangle}_{\tilde{h}_1[n]} \varphi_{1,n}(t)$$

<u>General:</u> Given <u>arbitrary</u> scaling and wavelet function φ and ψ, we can derive two filters:

conjugate mirror filters

Low-pass : $\tilde{h}_0[n] = \langle \varphi, \varphi_{1,n} \rangle$

High-pass : $\tilde{h}_1[n] = \langle \psi, \varphi_{1,n} \rangle$

Last Time:

$$V_1 = V_0 \oplus W_0$$

$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$$

$$\vdots$$

$$V_i = V_0 \oplus W_1 \oplus \cdots \oplus W_{i-1}$$

$$\vdots$$

$$L^2(\mathbb{R}) = V_0 \oplus \left(\bigoplus_{i=0}^{\infty} W_i \right)$$

Obs: $\underbrace{\{\varphi_{0,n}\}_{n\in\mathbb{Z}}}_{V_0} \cup \bigcup_{i=0}^{\infty} \underbrace{\{\psi_{i,n}\}_{n\in\mathbb{Z}}}_{W_i}$

is an <u>ortho basis</u> for $L^2(\mathbb{R})$.

Equivalently,

$$\{\varphi_{0,n}\}_{n\in\mathbb{Z}} \cup \{\varphi_{i,n}\}_{i\geq 0,\, n\in\mathbb{Z}}$$

is an ortho basis for $L^2(\mathbb{R})$.

Q: What does this mean for $f \in L^2(\mathbb{R})$?

A: f can be written as the sum of its projections onto $V_0, W_0, W_1, W_2, \ldots$

$$f(t) = \text{Proj}_{V_0} f(t) + \sum_{i=0}^{\infty} \text{Proj}_{W_i} f(t).$$

$$\underbrace{\text{Proj}_{V_0} f(t)}_{\text{coarse approximation}} = \sum_{n \in \mathbb{Z}} \underbrace{\langle f, \varphi_{0,n} \rangle}_{\substack{a_0[n] \\ \text{approx. coeffs.}}} \varphi_{0,n}(t)$$

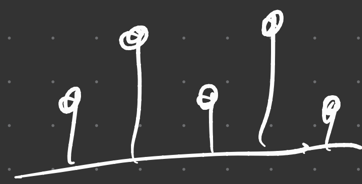
$$\underbrace{\text{Proj}_{W_i} f(t)}_{\text{details}} = \sum_{n \in \mathbb{Z}} \underbrace{\langle f, \psi_{i,n} \rangle}_{\substack{d_i[n] \\ \text{detail coeffs.}}} \psi_{i,n}(t)$$

$$f(t) = \sum_{n \in \mathbb{Z}} a_0[n] \varphi_{0,n}(t) + \sum_{i=1}^{\infty} \sum_{n \in \mathbb{Z}} d_i[n] \psi_{i,n}(t)$$

Synthesis Procedure

Analysis Procedure

Ex: finest resolution is $j=3$:



$$a_3[n] = \langle f, \varphi_{3,n} \rangle$$

$$\underset{V_3}{f_3(t)} = \sum_{n \in \mathbb{Z}} a_3[n]\, \varphi_{3,n}(t)$$

How do we get $a_2[n]$ & $d_2[n]$ from $a_3[n]$?

$$V_3 = V_2 \oplus W_2$$

project onto these subspaces

Project onto V_2

$$\text{Proj}_{V_2} f_3(t) = f_2(t) = \sum_{m \in \mathbb{Z}} \langle f_3, \varphi_{2,m} \rangle \varphi_{2,m}(t)$$

$$= \sum_{m \in \mathbb{Z}} \left\langle \sum_{n \in \mathbb{Z}} a_3[n] \varphi_{3,n}, \varphi_{2,m} \right\rangle \varphi_{2,m}(t)$$

$$= \sum_{m \in \mathbb{Z}} \underbrace{\left[\sum_{n \in \mathbb{Z}} a_3[n] \langle \varphi_{3,n}, \varphi_{2,m} \rangle \right]}_{a_2[m]} \varphi_{2,m}(t)$$

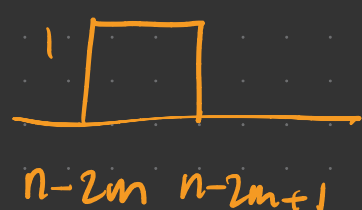
$$\langle \varphi_{3,n}, \varphi_{2,m} \rangle = \int_{-\infty}^{\infty} 2^{\frac{3}{2}} \varphi(2^3 t - n)\, 2\varphi(2^2 t - m)\, dt$$

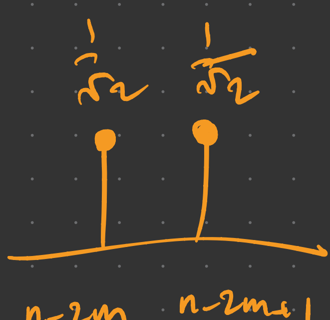
$$\begin{bmatrix} u = 2^3 t \\ du = 2^3\, dt \end{bmatrix}$$

$$= \frac{2^{\frac{5}{2}}}{2^3} \int_{-\infty}^{\infty} \varphi(u - n)\, \varphi\left(\frac{u}{2} - m\right) du$$

$$\begin{bmatrix} t = u - 2m \\ dt = du \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \varphi(t - (n - 2m))\, \varphi\left(\frac{t}{2}\right) dt$$

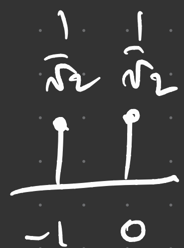




this integral is zero unless $n-2m = 0$ or $n-2m=1$ in which case it is 1.

$$= \tilde{h}_0[n-2m]$$

$$f_2(t) = \sum_{m \in \mathbb{Z}} \underbrace{\left(\sum_{n \in \mathbb{Z}} a_3[n]\, \tilde{h}_0[n-2m] \right)}_{a_2[m]} \varphi_{2,m}(t)$$



Define $h_0[k] = \tilde{h}_0[-k]$ time reversal

$$\Rightarrow a_2[m] = \sum_{n \in \mathbb{Z}} a_3[n]\, h_0[2m-n]$$

$$= (a_3 * h_0)[2m]$$ downsampling

$$a_3[n] \longrightarrow \boxed{H_0} \longrightarrow \boxed{\downarrow 2} \longrightarrow a_2[n]$$

Project onto W_2

$$\mathrm{Proj}_{W_2} f_3(t) = g_2(t) = \sum_{m \in \mathbb{Z}} \langle f_3, \psi_{2,m} \rangle \psi_{2,m}(t)$$

$$= \sum_{m \in \mathbb{Z}} \left\langle \sum_{n \in \mathbb{Z}} a_3[n] \varphi_{3,n}, \psi_{2,m} \right\rangle \psi_{2,m}(t)$$

$$= \sum_{m\in\mathbb{Z}} \underbrace{\left[\sum_{n\in\mathbb{Z}} a_3[n] \langle \varphi_{3,n}, \psi_{2,m}\rangle\right]}_{d_2[m]} \psi_{2,m}(t)$$

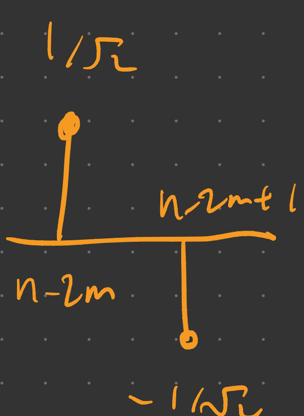
$$\langle \varphi_{3,n}, \psi_{2,m}\rangle = \int_{-\infty}^{\infty} 2^{\frac{3}{2}} \varphi(2^3 t - n)\, 2\psi(2^2 t - m)\, dt$$

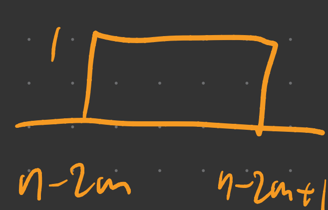
$$\begin{bmatrix} u = 2^3 t \\ du = 2^3 dt \end{bmatrix}$$

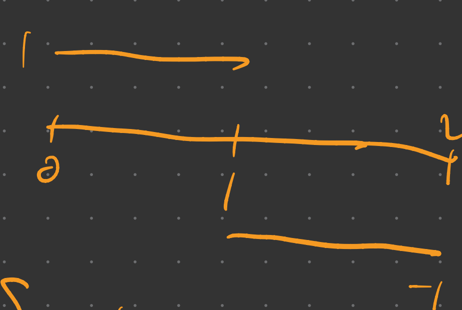
$$= \frac{2^{\frac{5}{2}}}{2^3} \int_{-\infty}^{\infty} \varphi(u-n)\, \psi\left(\frac{u}{2} - m\right) dt$$

$$\begin{bmatrix} t = u - 2m \\ dt = du \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \varphi(t - (n-2m))\, \psi\left(\frac{t}{2}\right) dt$$





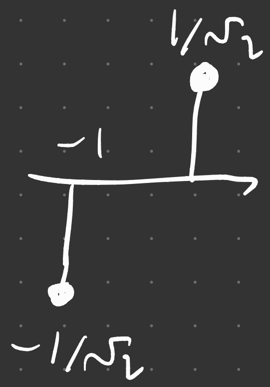


$n-2m = 0 \Rightarrow \delta = 1$

$n-2m = 1 \Rightarrow \delta = -1$

$$= \tilde{h}_1[n-2m]$$

$$g_2(t) = \sum_{m\in\mathbb{Z}} \underbrace{\left(\sum_{n\in\mathbb{Z}} a_3[n]\, \tilde{h}_1[n-2m] \right)}_{d_2[m]} \psi_{2,m}(t)$$



Define $h_1[n] = \tilde{h}_1[-n]$

$$\Rightarrow d_2[m] = \sum_{n\in\mathbb{Z}} a_3[n]\, h_1[2m-n]$$

downsampling

$$= (a_3 * h_1)[2m]$$

$$a_3[n] \longrightarrow \boxed{H_1} \longrightarrow \boxed{\downarrow 2} \longrightarrow d_2[n]$$

Obs: Given an approximation $a_i[n]$, we can get all other coefficients $a_k[n]$, $k \le i$, and $d_\ell[n]$, $\ell \le i$, extremely fast: Fast Wavelet Transform

Concrete implementation: DWT

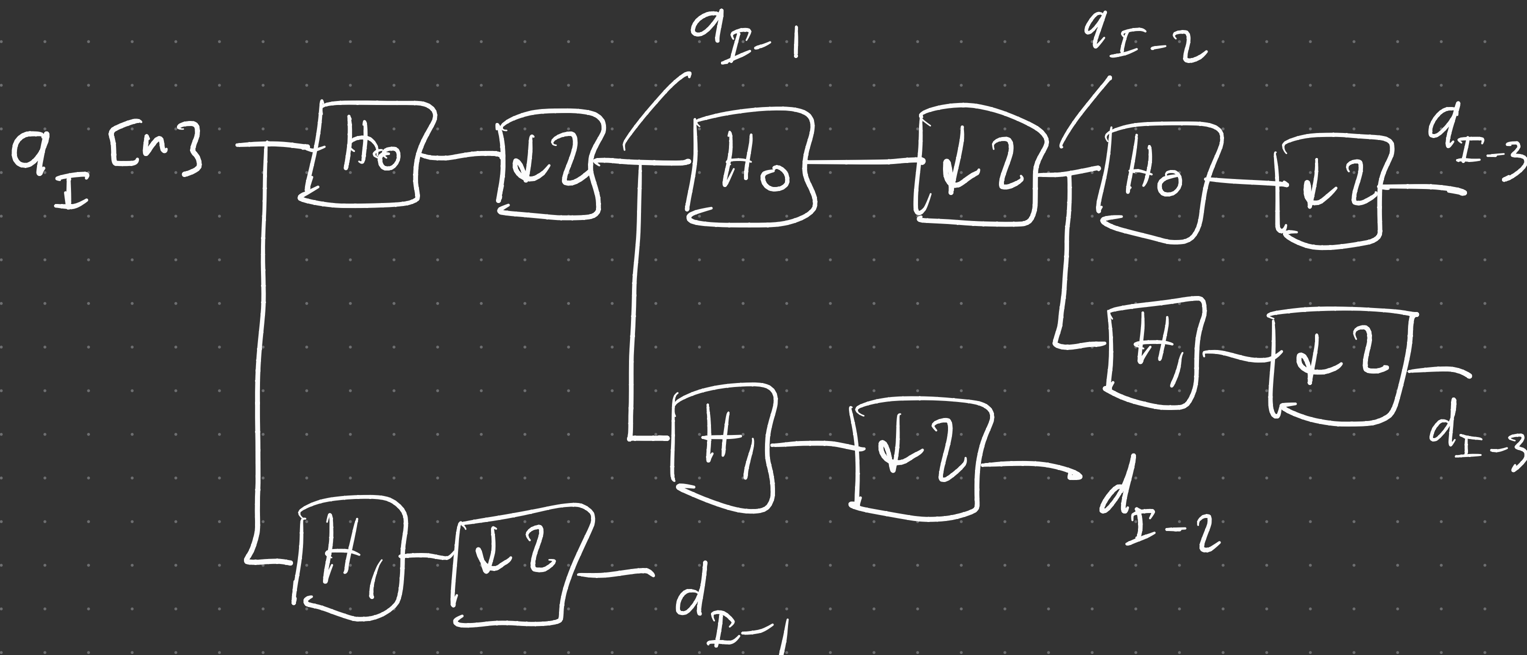
General DWT Pipeline:

$f(t)$: underlying analog signal

$a_I[n]$: discrete approximation at some <u>high resolution</u> I that you actually get to work with (e.g., image, audio, etc.)

$$\boxed{f(t) \approx \sum_{n \in \mathbb{Z}} a_I[n]\, \varphi_{I,n}(t)}$$

- Perform a DWT:



- Do some (nonlinear) processing on, e.g.,

$$a_{I-3},\ d_{I-3},\ d_{I-2},\ d_{I-1}$$

- Perform an IDWT on the processed signals, resulting in $\hat{a}_I[n]$

- Processed analog signal:

$$\hat{f}(t) = \sum_{n \in \mathbb{Z}} \hat{a}_I[n]\, \varphi_{I,n}(t)$$

"Think analog, act digital"

MRA DWT