Last Time: General Wavelet Analysis Pipeline

Setup:

$f(t)$: underlying analog signal

$a_I[n]$: discrete approximation at some high resolution I that you actually get to work with (e.g., image, audio signal, etc.)

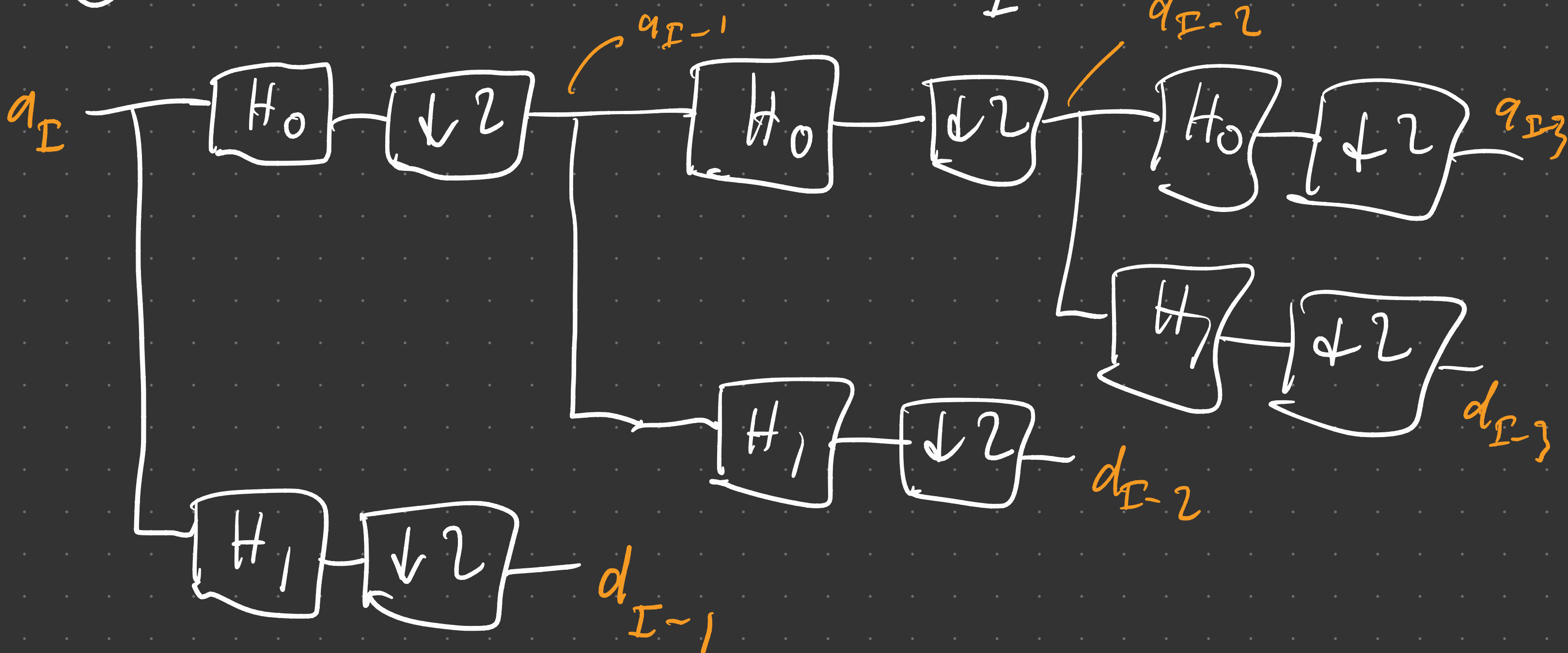
$\varphi(t), \psi(t)$: scaling and wavelet functions that give rise to a MRA.

Fundamental assumption:

$$\boxed{f(t) \approx \sum_{n \in \mathbb{Z}} a_I[n] \varphi_{I,n}(t)}$$

Pipeline:

① Perform a DWT on $a_I[n]$



→ Iterate as many times as you want

→ The filters come from the scaling and wavelet functions

- $h_0[n] = \tilde{h}_0[-n]$, $\tilde{h}_0[n] = \langle \varphi, \varphi_{1,n} \rangle$
- $h_1[n] = \tilde{h}_1[-n]$, $\tilde{h}_1[n] = \langle \psi, \varphi_{1,n} \rangle$

② Do some (typically nonlinear) processing on, e.g.,

$$a_{I-3}, d_{I-3}, d_{I-2}, d_{I-1}$$

→ For thresholding procedures people typically only threshold the detail coefficients

3. Perform an IDWT on the processed signals, resulting in $\hat{a}_I[n]$.

4. Processed analog signal:

$$\hat{f}(t) = \sum_{n \in \mathbb{Z}} \hat{a}_I[n] \, \varphi_{I,n}(t)$$

Slogan: "Think analog, act digital"
MRA DWT

Obs: This whole pipeline is "blind" to the underlying MRA, φ, and ψ.

- We only need the filters h_0 & h_1 (or, equivalently $\tilde{h}_0$ & $\tilde{h}_1$).

Remark: Last time we saw how φ & ψ determine $\tilde{h}_0$ & $\tilde{h}_1$. Today we will see how $\tilde{h}_0$ & $\tilde{h}_1$ determine φ & ψ.

Last Time: "filters from wavelets"

Today: "wavelets from filters"

Idea: We can iterate the DWT as many times as we want. Therefore, we should be able to iterate it infinitely many times.

$\boxed{\text{This procedure needs to converge}}$

Equivalently, given $\tilde{h}_0[n]$, the iteration

cascade algorithm

$$\varphi^{(k+1)}(t) = \sum_{n \in \mathbb{Z}} \underbrace{\tilde{h}_0[n]}_{h[n]} \sqrt{2}\, \varphi^{(k)}(2t-n)$$

must converge.

Obs: We are looking for the fixed point of the two-scale equation.

Def$^{\underline{n}}$: x is called the fixed point of

$$x_{k+1} = F(x_k)$$

if $x = F(x)$.

Obs: Given an initial value x_0, if the sequence $\{x_k\}_{k=0}^{\infty}$ converges to x, i.e., $\lim_{k\to\infty} x_k = x$, then x is a fixed point of F.

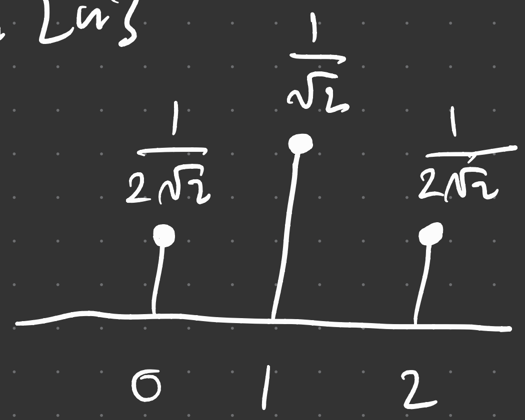
Ex: Does the sequence $x_{k+1} = \frac{x_k}{2}$ converge? What is its fixed point?

$$x_1 = \frac{x_0}{2}, \quad x_2 = \frac{x_1}{2} = \frac{x_0}{4}, \quad \dots, \quad \boxed{x_k = \frac{x_0}{2^k}}$$

This sequence converges to 0 for any choice of x_0.

$$0 = \frac{0}{2}$$ fixed point

<u>Exer:</u> $h[n]$



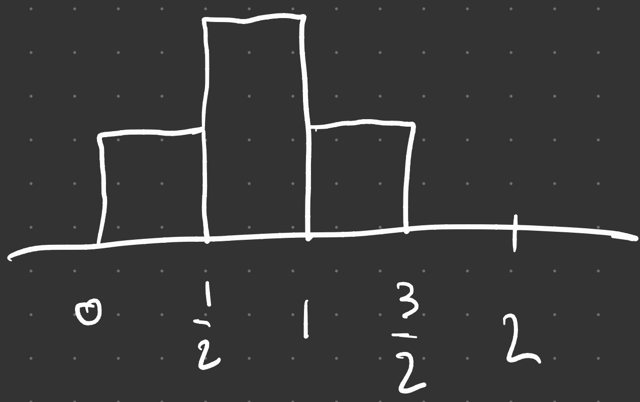
What is the fixed point of the cascade algorithm?

$$\boxed{\varphi(t) = \frac{1}{2}\varphi(2t) + \varphi(2t-1) + \frac{1}{2}\varphi(2t-2)}$$

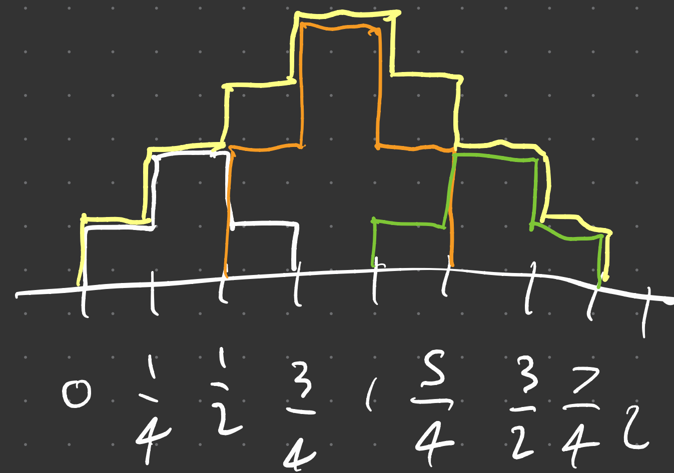
Let us try to run the cascade algorithm with the initial condition

$\varphi^{(0)}(t)$

$\varphi^{(1)}(t)$

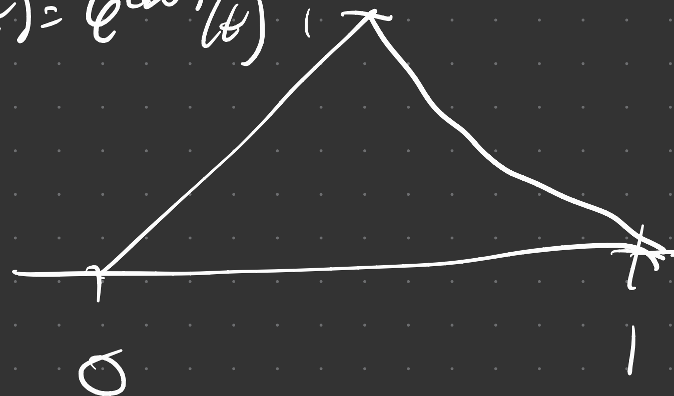


$\varphi^{(2)}(t)$



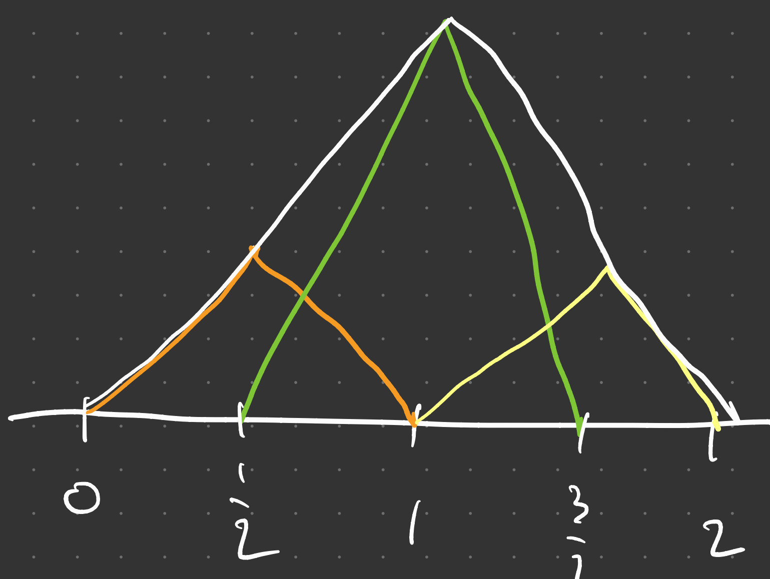
keep iterating ...

$\varphi(t) = \varphi^{(\infty)}(t)$



<u>Exer:</u> Do one more iteration.

<u>Check:</u>



Q: How do we understand this in general?

A: Fourier transform.

$$\varphi(t) = \sum_{n\in\mathbb{Z}} h[n]\,\sqrt{2}\,\varphi(2t-n)$$

$\mathcal{F}$

$$\underbrace{\int_{-\infty}^{\infty}\varphi(t)\,e^{-j\omega t}\,dt}_{\Phi(\omega)} = \int_{-\infty}^{\infty}\left(\sum_{n\in\mathbb{Z}} h[n]\sqrt{2}\,\varphi(2t-n)\right)e^{-j\omega t}\,dt$$

$$= \sum_{n\in\mathbb{Z}} h[n]\cdot\sqrt{2}\int_{-\infty}^{\infty}\varphi(2t-n)\,e^{-j\omega t}\,dt$$

$$\begin{bmatrix} u = 2t-n \\ du = 2\,dt \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} h[n]\int_{-\infty}^{\infty}\varphi(u)\,e^{-j\omega\left(\frac{u+n}{2}\right)}\,dt$$

$$= \underbrace{\frac{1}{\sqrt{2}}\left(\sum_{n\in\mathbb{Z}} h[n]\, e^{-j\omega\frac{n}{2}}\right)}_{H\left(e^{j\frac{\omega}{2}}\right)} \underbrace{\int_{-\infty}^{\infty} \phi(u)\, e^{-j\omega\frac{u}{2}}\, dt}_{\Phi\left(\frac{\omega}{2}\right)}$$

$$\Phi(\omega) = \frac{H\left(e^{j\frac{\omega}{2}}\right)}{\sqrt{2}}\, \Phi\left(\frac{\omega}{2}\right)$$

$$= \frac{H\left(e^{j\frac{\omega}{2}}\right)}{\sqrt{2}} \cdot \frac{H\left(e^{j\frac{\omega}{4}}\right)}{\sqrt{2}}\, \Phi\left(\frac{\omega}{4}\right)$$

$$= \frac{H\left(e^{j\frac{\omega}{2}}\right)}{\sqrt{2}} \cdot \frac{H\left(e^{j\frac{\omega}{4}}\right)}{\sqrt{2}} \cdot \frac{H\left(e^{j\frac{\omega}{8}}\right)}{\sqrt{2}}\, \Phi\left(\frac{\omega}{8}\right)$$

$$\vdots$$

$$= \Phi(0) \prod_{i=1}^{\infty} \frac{H\left(e^{j2^{-i}\omega}\right)}{\sqrt{2}}$$

Recall: $\Phi(0) = \int_{-\infty}^{\infty} \varphi(t) \cancel{e^{-j0t}}^{=1} \, dt$

$$= \int_{-\infty}^{\infty} \varphi(t) \, dt$$

We often impose that $\int \varphi(t) \, dt = 1$

(e.g., this holds for the box )

Infinite-Product formula:

$$\Phi(\omega) = \prod_{i=1}^{\infty} \frac{H(e^{j2^{-i}\omega})}{\sqrt{2}}$$

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\omega) e^{j\omega t} \, d\omega$$

inverse Fourier transform

"wavelets from filters"

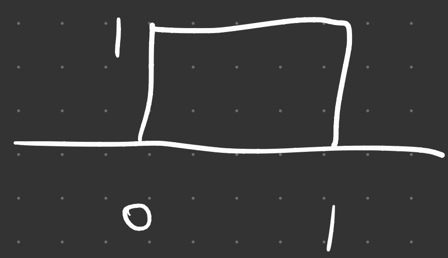
Obs: Convergence of this infinite product is the same as convergence of the cascade algorithm, which is the same as convergence of an *infinite-level* DWT.

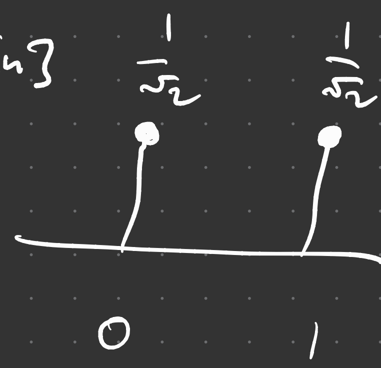
Remark: If $H(e^{j\omega})$ is continuously differentiable at $\omega = 0$, and

$$\min_{\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]} |H(e^{j\omega})| > 0,$$

then the infinite product converges. (Mallat, 1989).

Exer: $\varphi(t)$ $h[n]$





Compute $\Phi(\omega)$ from the infinite product formula.

$$\frac{H(e^{j\omega})}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(1+e^{-j\omega}\right)\right]$$

$$= \frac{1}{2}\left(1+e^{-j\omega}\right)$$

$$H^{(N)}(e^{j\omega}) = \frac{1}{2^N}\underbrace{\prod_{i=1}^{N}\left(1+e^{-j2^{-i}\omega}\right)}_{2^N \text{ terms}}$$

[Check at home]

$$= \frac{1}{2^N}\sum_{k=0}^{2^N-1} e^{-j2^{-N}\omega k}$$

$$\left[\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}\right] = \frac{1}{2^N}\left(\frac{1-e^{-j\omega}}{1-e^{-j2^{-N}\omega}}\right)$$

$$\left[e^{-j\theta} = 1 - i\theta + \frac{\theta^2}{2} - \frac{i\theta^2}{3} + \cdots\right]$$

$$\theta = 2^{-N}\omega$$

$$\xrightarrow[N\to\infty]{} \frac{1-e^{-j\omega}}{j\omega}$$

$$\Phi(\omega) = \frac{1-e^{-j\omega}}{j\omega} = e^{-j\frac{\omega}{2}}\left(\frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{j\omega}\right)$$

modulated sinc

$$= e^{-j\frac{\omega}{2}}\left(\frac{2\sin\left(\frac{\omega}{2}\right)}{\omega}\right)$$

Remark: This recovers Euler's celebrated infinite product formula for the sinc.

Fundamental Theorem of Wavelet Analysis (Mallat, 1989)

Let $\varphi \in L^2(\mathbb{R})$ be a valid scaling function. Then, the filter $h[n] = \langle \varphi, \varphi_{1,n} \rangle$ must satisfy

① $$|H(e^{j\omega})|^2 + |H(e^{j(\omega+\pi)})|^2 = 2$$

② $$H(e^{j0}) = \sum_{n\in\mathbb{Z}} h[n] = \sqrt{2}$$

③ $$\min_{\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]} |H(e^{j\omega})| > 0.$$

On the other hand, given a filter $h[n]$ such that $H(e^{j\omega})$ satisfies ①, ②, & ③, then the inverse Fourier transform of

$$\Phi(\omega) = \prod_{i=1}^{\infty} \frac{H(e^{j2^{-i}\omega})}{\sqrt{2}}$$

exists and is a valid scaling function.

Obs: This gives a one-to-one correspondance between scaling functions φ and filters h.

Remark:

$$\tilde{h}_0[n] = h[n]$$

if you allow for complex coeffs.

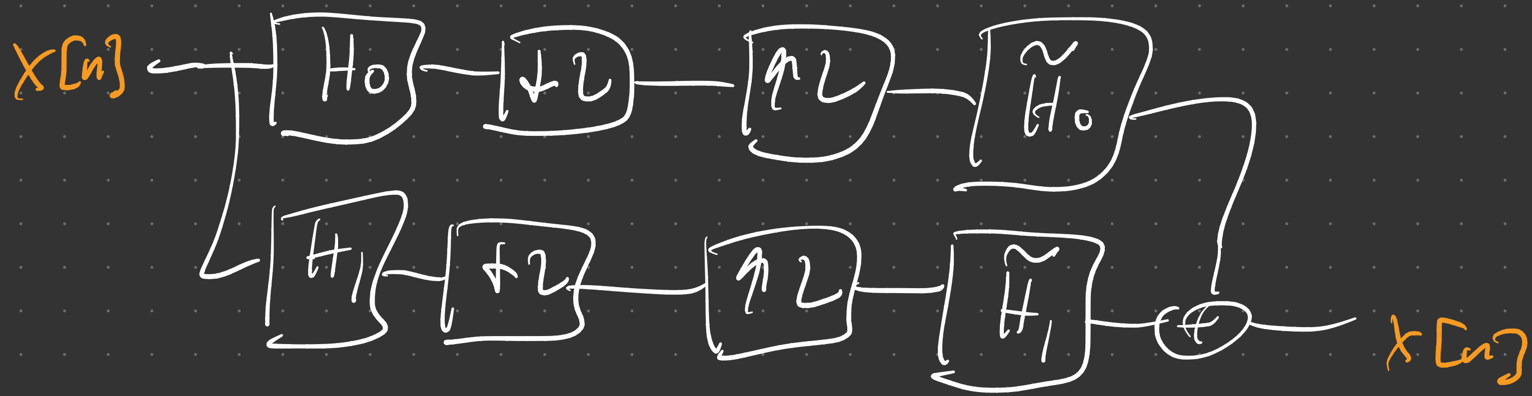
$$\tilde{h}_1[n] = (-1)^{1-n} h^*[1-n]$$

conjugate mirror filters

Ex: Haar

Ex: $h_0[n] = \tilde{h}_0[-n]$

$h_1[n] = \tilde{h}_1[-n]$



Show that this is a PR FB with $L=0$.

$$\tilde{H}_0(z)\,H_0(z) + \tilde{H}_1(z)\,H_1(z) = 2\,\cancel{z^{-L}} \nearrow 1$$

$$H(e^{j\omega})\,H(e^{-j\omega}) + H\left(e^{j(\omega+\pi)}\right)H\left(e^{-j(\omega+\pi)}\right) = 2$$

$$|H(e^{j\omega})|^2 + |H(e^{j(\omega+\pi)})|^2 = 2$$

→ This is ① from the fundamental theorem.