Last Time: Fundamental Theorem of Wavelet Analysis

Let $\varphi \in L^2(\mathbb{R})$ be a valid scaling function.

Then, the filter $h[n] = \langle \varphi, \varphi_{1,n} \rangle$ must satisfy

① $|H(e^{j\omega})|^2 + |H(e^{j(\omega+\pi)})|^2 = 2$

② $H(e^{j0}) = \sum_{n\in\mathbb{Z}} h[n] = \sqrt{2}$

③ $\min_{\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]} |H(e^{j\omega})| > 0.$

On the other hand, given a filter $h[n]$ such that $H(e^{j\omega})$ satisfies ①, ②, & ③, then the inverse Fourier transform of

$$\Phi(\omega) = \prod_{i=1}^{\infty} \frac{H(e^{j2^{-i}\omega})}{\sqrt{2}}$$

exists and is a valid scaling function:

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(\omega)\, e^{j\omega t}\, d\omega$$

Obs: There is a one-to-one correspondence between scaling functions and low-pass filters.

Conjugate mirror filters

$$\tilde{h}_0[n] = h[n]$$

~~complex coefficients~~

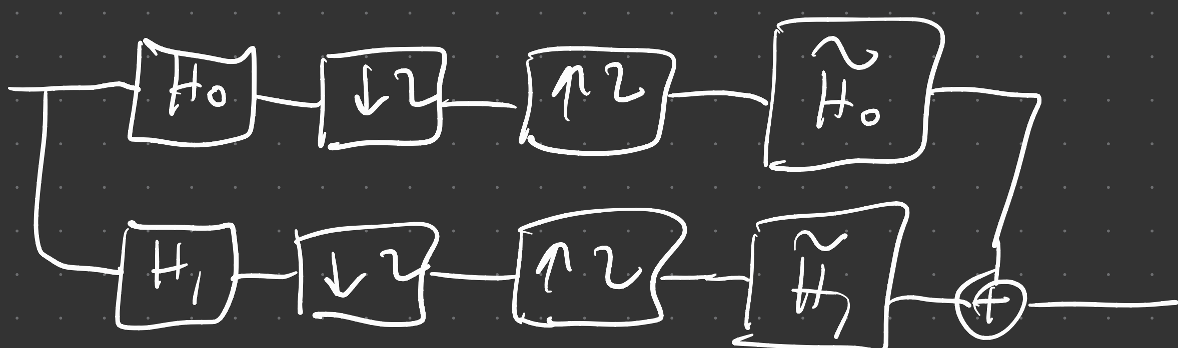
$$\tilde{h}_1[n] = (-1)^{1-n}\, h^*[1-n]$$

focus on real in this class

$$h_0[n] = \tilde{h}_0[-n]$$

$$h_1[n] = \tilde{h}_1[-n]$$

The two channel FB:



is PR.

Exer: Determine the delay of the system.

$$\underbrace{\tilde{H}_0(z)\,H_0(z) + \tilde{H}_1(z)\,H_1(z)}_{\text{distortion}} = 2z^{-L} \leftarrow \text{delay}$$

$$\tilde{H}_0(e^{j\omega})\,H_0(e^{j\omega}) + \tilde{H}_1(e^{j\omega})\,H_1(e^{j\omega}) = 2e^{-jL\omega}$$

2π-periodic

$$\tilde{H}_0(e^{j\omega}) = H(e^{j\omega})$$

$$\tilde{H}_1(e^{j\omega}) = e^{-j\omega}\,H(e^{-j(\omega+\pi)}) = e^{-j\omega}\,H(e^{-j(\omega-\pi)})$$

$$H_0(e^{j\omega}) = H(e^{-j\omega})$$

$$H_1(e^{j\omega}) = e^{j\omega}\,H(e^{j(\omega+\pi)})$$

$$\rightarrow H(e^{j\omega})\,H(e^{-j\omega}) + \cancel{e^{-j\omega}}\,H(e^{-j(\omega+\pi)})\,\cancel{e^{j\omega}}\,H(e^{j(\omega+\pi)})$$

$$= |H(e^{j\omega})|^2 + |H(e^{j(\omega+\pi)})|^2$$

$$= 2$$ [by the fundamental theorem]

$$\Longrightarrow \boxed{L=0}$$

Q: Why are we doing any of this?

Why not just use FFTs?

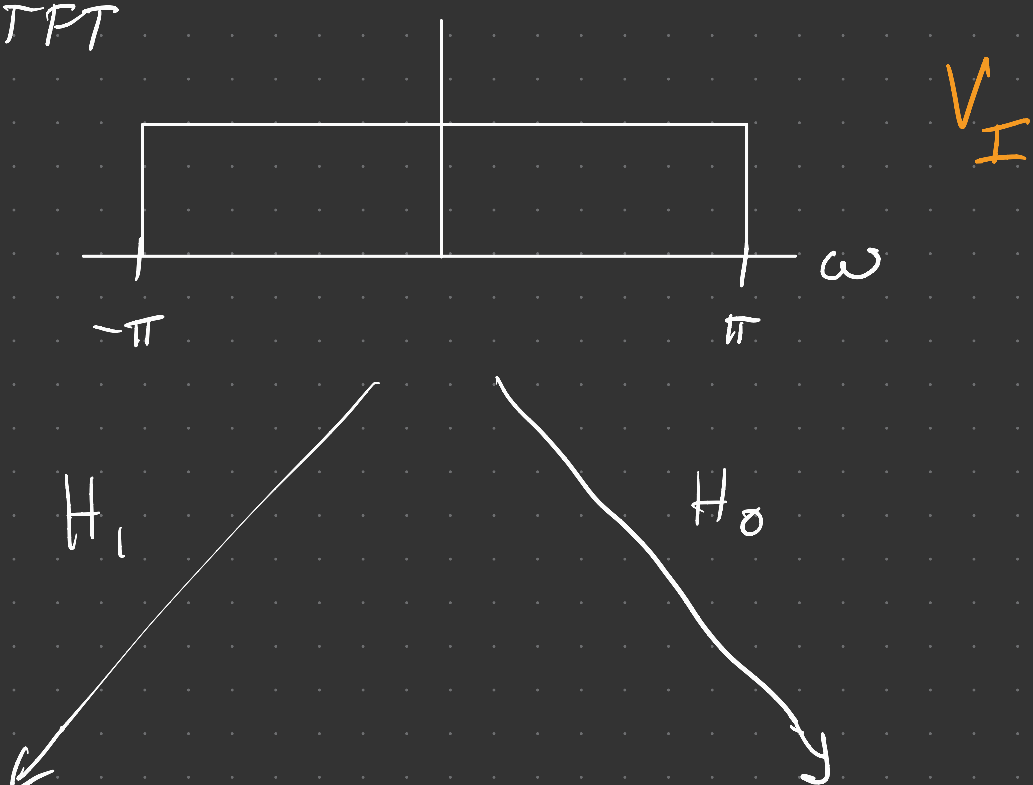
A: DWTs have a "nicer" frequency-band decomposition.

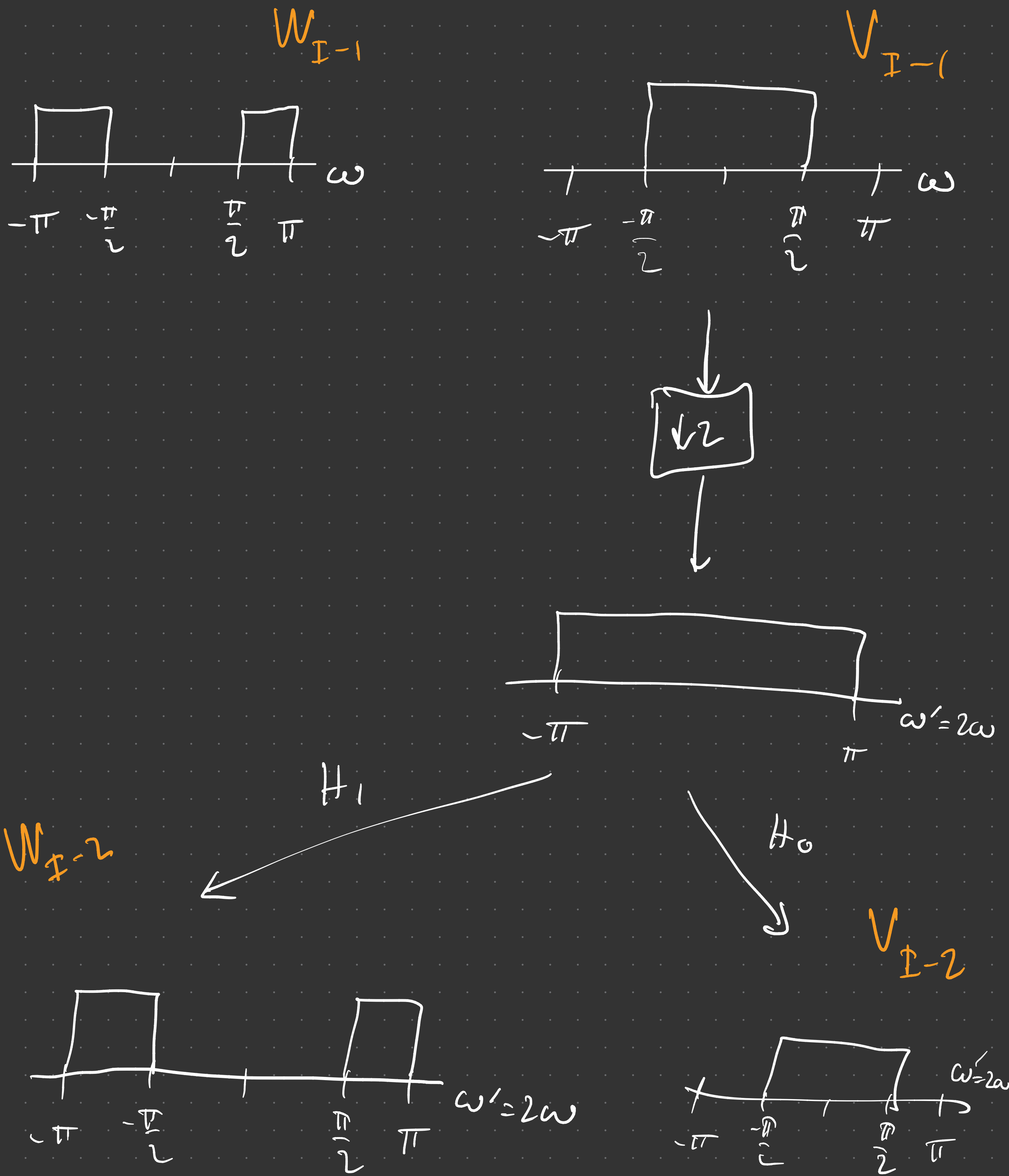
Recall:

$$x[n] \longrightarrow \boxed{\downarrow 2} \longrightarrow x[2n] = y[n]$$

$$Y(e^{j\omega}) = \frac{1}{2}\left(X\left(e^{j\frac{\omega}{2}}\right) + X\left(e^{j\left(\frac{\omega-2\pi}{2}\right)}\right)\right)$$

Suppose we have a discrete approx $a_I[n]$ with DTFT





W_{I-1}
ω
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
V_{I-1}
ω
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\downarrow 2$
$-\pi$
π
$\omega' = 2\omega$
H_1
H_0
W_{I-2}
$\omega' = 2\omega$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
V_{I-2}
$\omega' = 2\omega$
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π

Frequency - Band Decomposition of the DWT

$$V_I \approx \omega \in [-\pi, \pi]$$

$$W_{I-1} \approx \omega \in \left[-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$$

$$V_{I-1} \approx \omega \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$W_{I-2} \approx \omega' \in \left[-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]$$

$$\equiv \omega \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right] \cup \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$$

$$V_{I-2} \approx \omega' \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\equiv \omega \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

$$W_{I-k} \approx \omega \in \left[-\frac{\pi}{2^{k-1}}, -\frac{\pi}{2^k}\right] \cup \left[\frac{\pi}{2^{k-1}}, \frac{\pi}{2^k}\right]$$

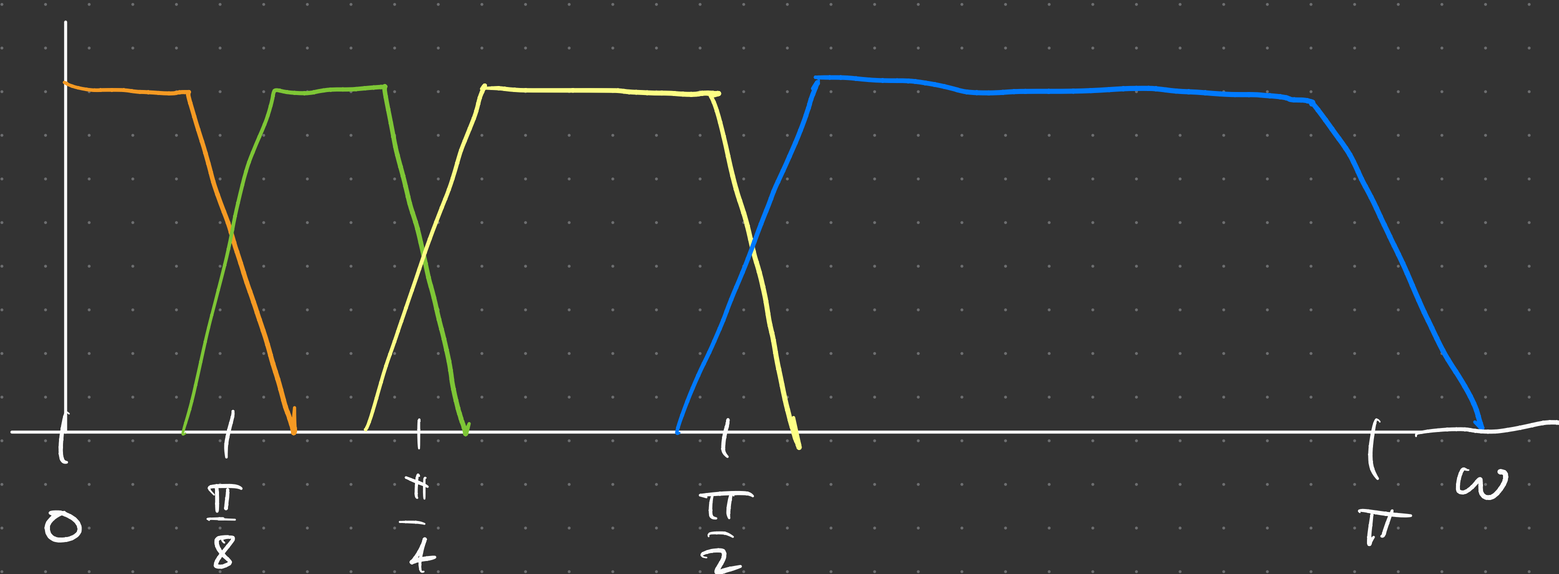
$$V_{I-k} \approx \omega \in \left[-\frac{\pi}{2^k}, \frac{\pi}{2^k}\right]$$

Obs: Wavelet spaces are (approximately) bandpass subspaces.

Obs: We have a logarithmic (base 2) set of bandwidths.

Remarks: The logarithmic frequency decomp. is similar to the octave decomp. in musical scales and is related to the response characteristics of the human ear.

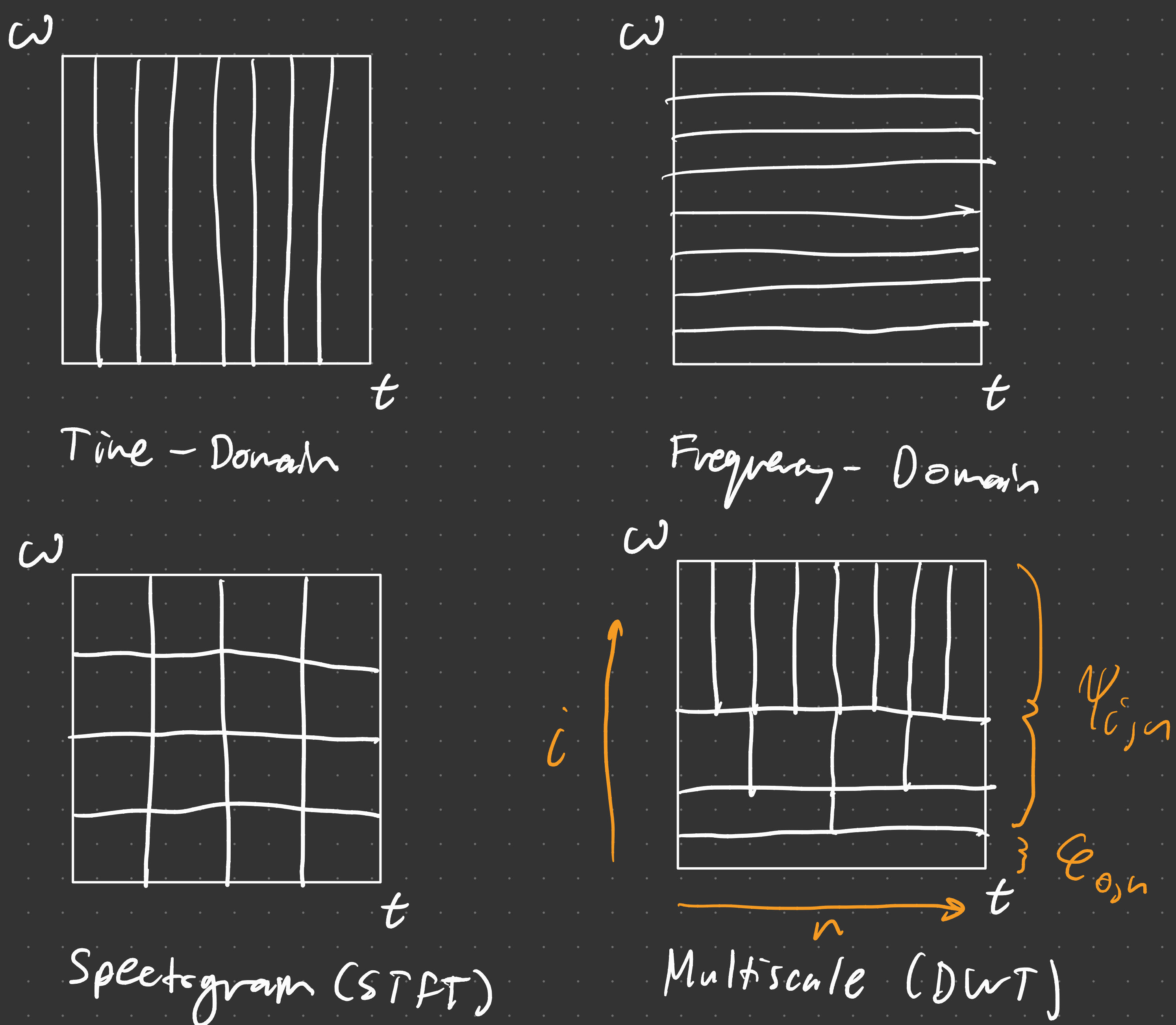
$$V_{I-3} \oplus W_{I-3} \oplus W_{I-2} \oplus W_{I-1} = V_I$$



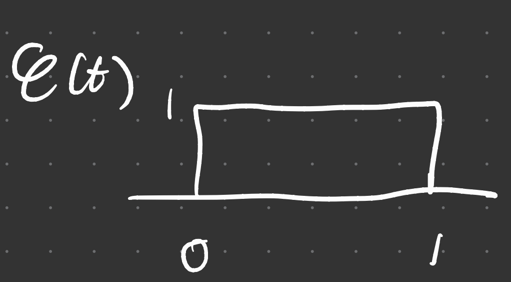
Obs: • High-frequency information is captured in short time instants.

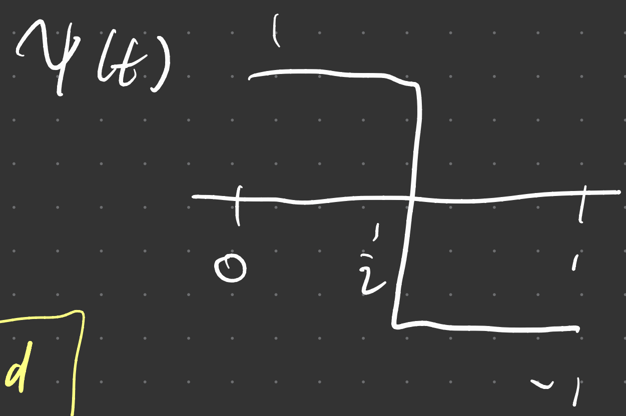
• Low-frequency information is captured in long time instants.

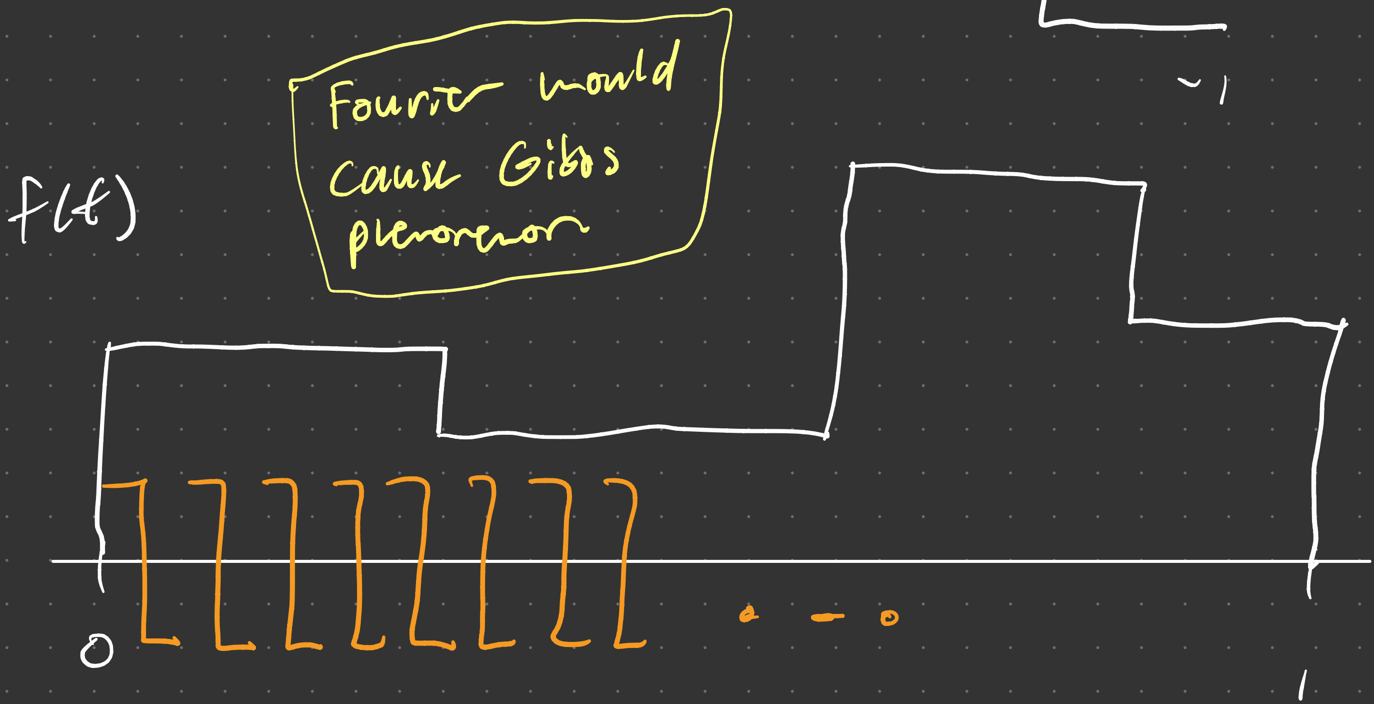
Time-Frequency Tilings



Haar Wavelets:







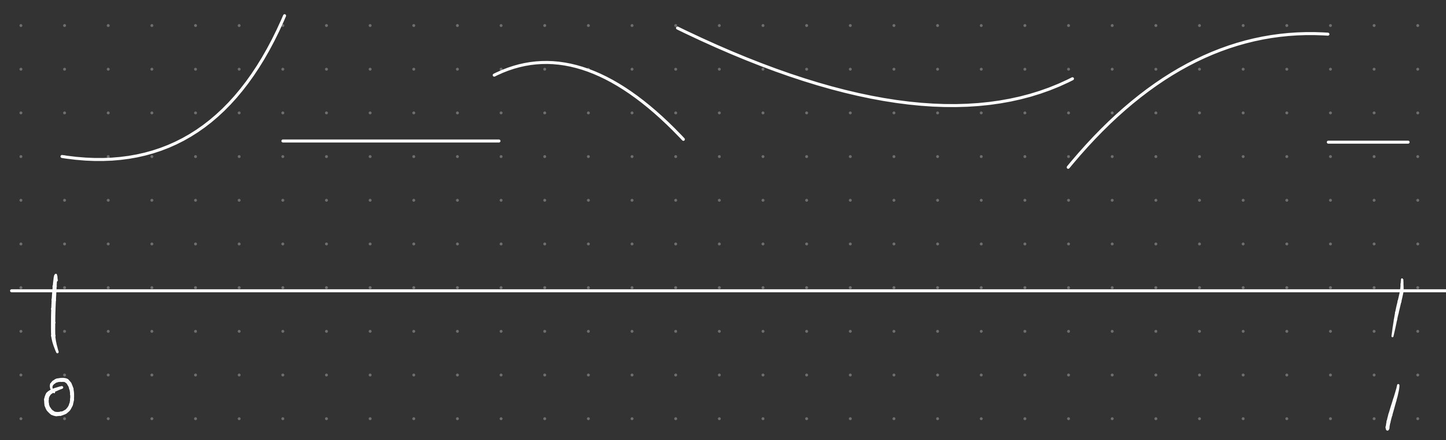
<u>Obs:</u> <u>All</u> wavelet coeffs will be zero <u>except</u> at the jumps.

<u>Q:</u> What about the scaling coefficients?

<u>A:</u> Only store <u>one</u> number (average of signal).

<u>Key Property:</u> $\int_{-\infty}^{\infty} \psi(t)\, dt = 0$ [one vanishing moment]

Remark: Real-life signals are approximately piecewise polynomial.



Q: Can we have higher-order wavelets?

yes:

A: Daubechies, symmlet, spline wavelets, etc.

Defn: A wavelet $\psi(t)$ is said to have p-vanishing moments if it satisfies

$$\int_{-\infty}^{\infty} t^m \psi(t)\, dt = 0$$

for all $m = 0, 1, \ldots, p-1$.

Remark: The number of vanishing moments is tightly linked to the support of the wavelet and the DWT filters.

Theorem (Daubechies, 1988):

A wavelet ψ with p-vanishing moments must have support at least $2p-1$, i.e., the length of

Closure

$$\operatorname{supp} \psi = \overline{\{ t \in \mathbb{R} : \psi(t) \neq 0 \}}$$

is at least $2p-1$.

Q: How are the # of vanishing moments related to the DWT filters?

A: # of zeros @ π of low-pass filters.

Proof: Theorem 7.4 in the book.

Q: • Which wavelets have the shortest support for a given # of vanishing moments?

• Which filters have the most # of zeros @ π for a given order?

A: Daubechies wavelets / filter

Here are pictures of some of the scaling functions ($N = 2p$ in the captions below):

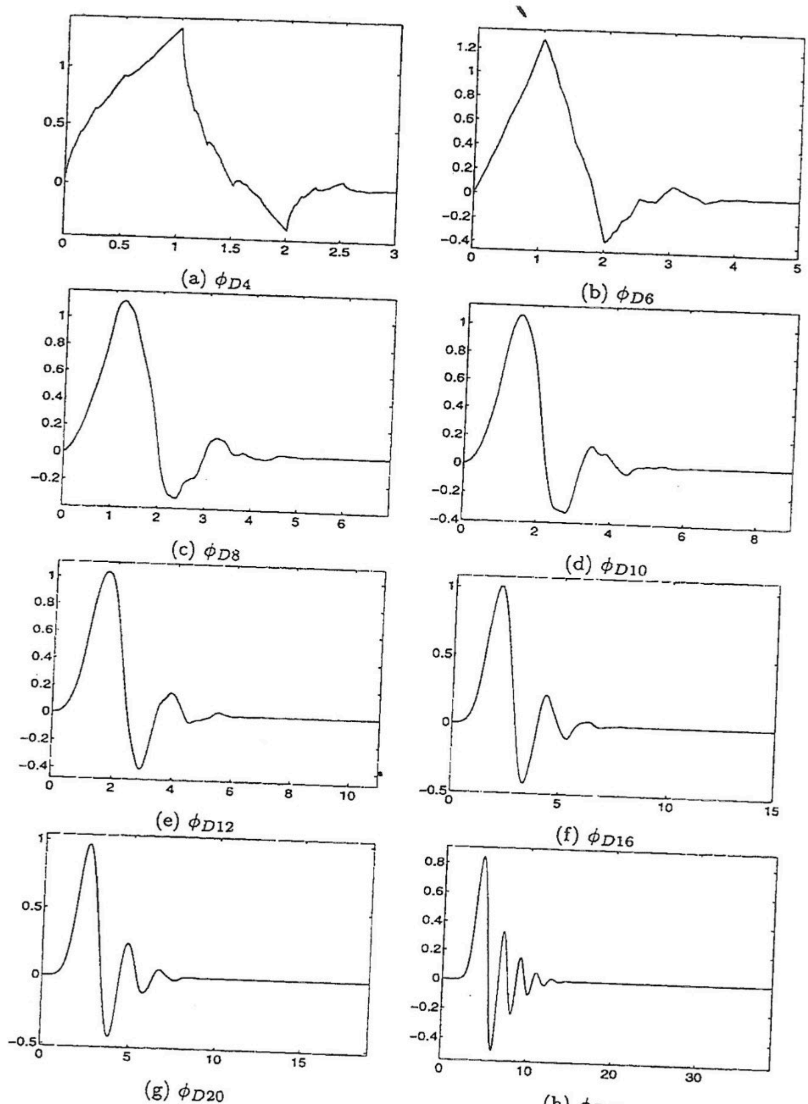


Figure 6.1. Daubechies Scaling Functions, $N = 4, 6, 8, \ldots, 40$

From Burrus et al, *Introduction to Wavelets* ...

Here are pictures of some of the wavelet functions ($N = 2p$ in the captions below):

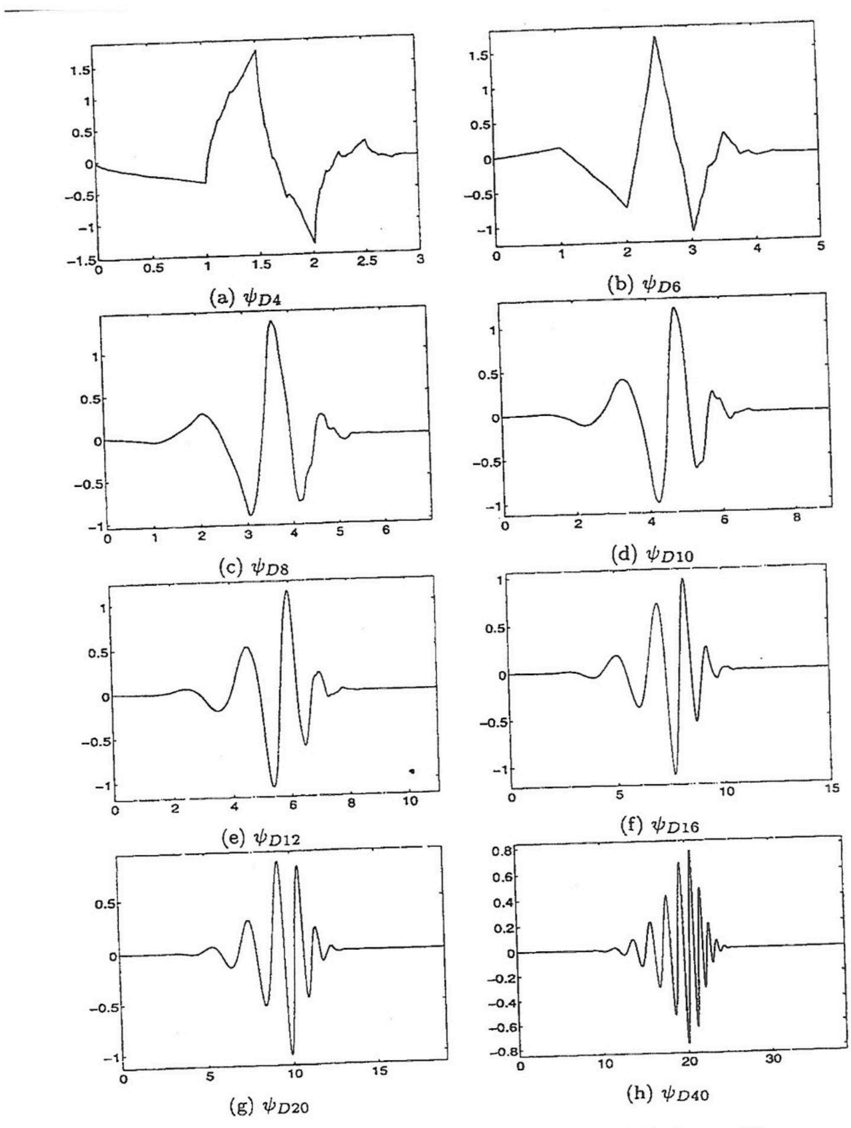


Figure 6.2. Daubechies Wavelets, $N = 4, 6, 8, \ldots, 40$

From Burrus et al, *Introduction to Wavelets ...*

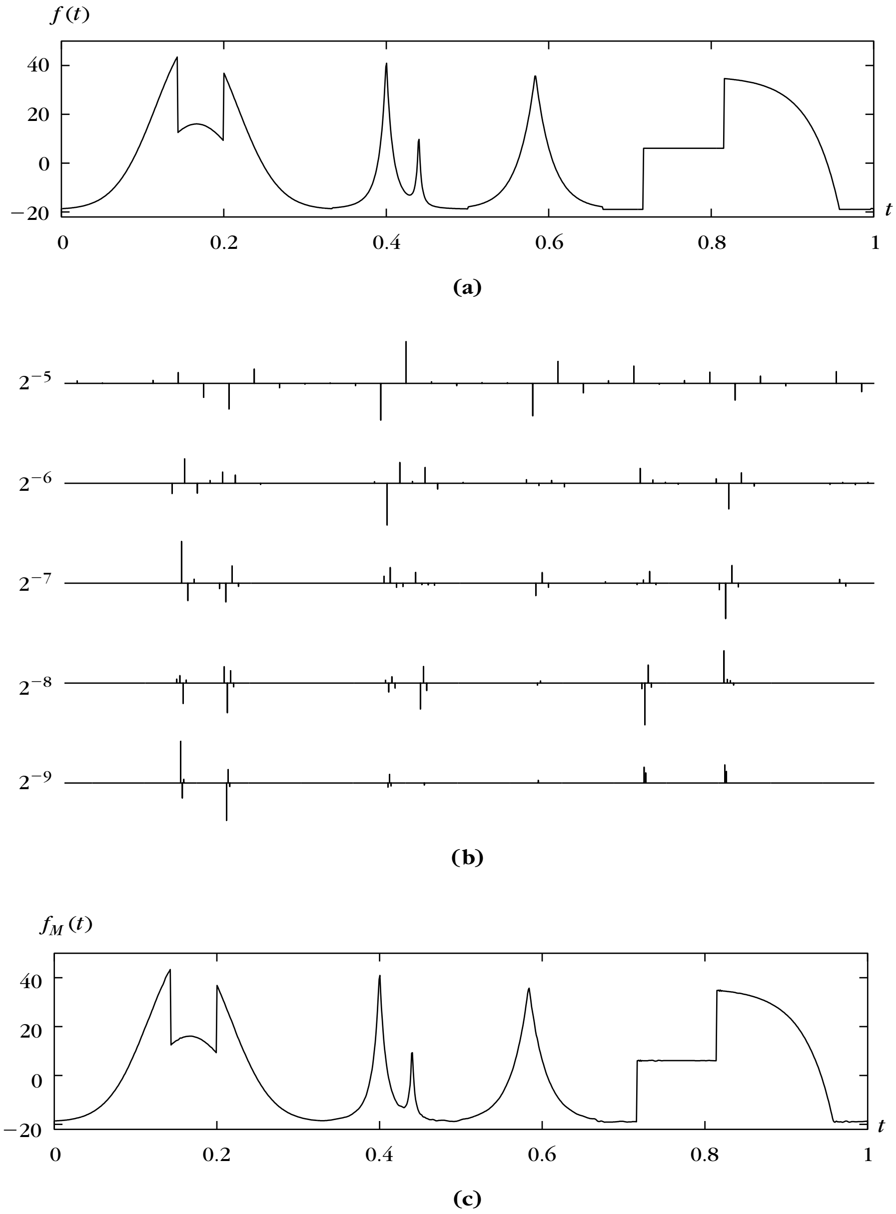


FIGURE 9.2

(a) Original signal f. **(b)** Each Dirac corresponds to one of the largest $M = 0.15\,N$ wavelet coefficients, calculated with a symmlet 4. **(c)** Nonlinear approximation f_M recovered from the M largest wavelet coefficients shown in (b), $\|f - f_M\|/\|f\| = 5.1\,10^{-3}$.

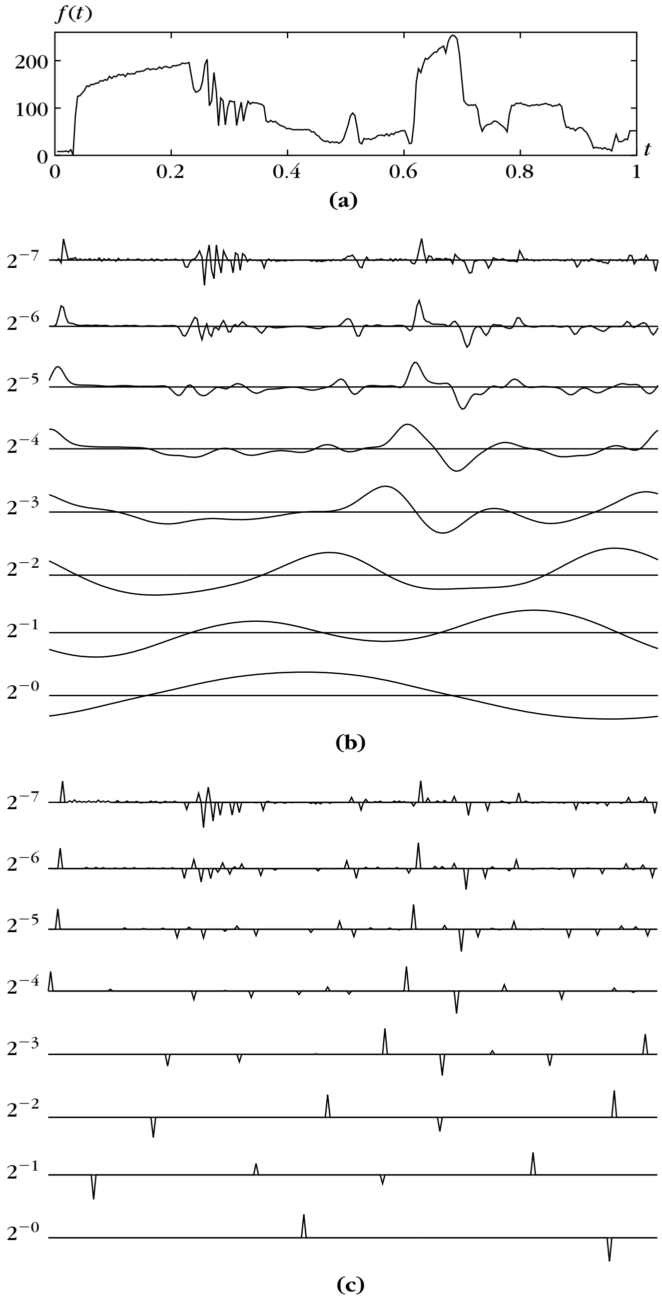


FIGURE 6.7

(a) Intensity variation along one row of the Lena image. **(b)** Dyadic wavelet transform computed at all scales $2N^{-1} \leqslant 2^j \leqslant 1$, with the quadratic spline wavelet $\psi = -\theta'$ shown in Figure 5.3. **(c)** Modulus maxima of the dyadic wavelet transform.