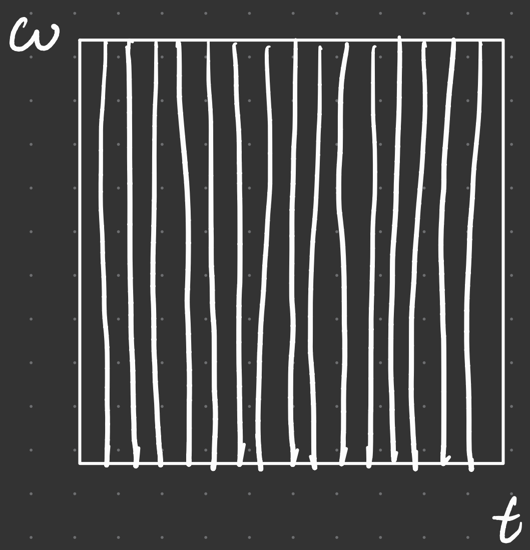
Last Time: Why?

- Wavelets have "nicer" time-frequency tilings compared to other signal representations.

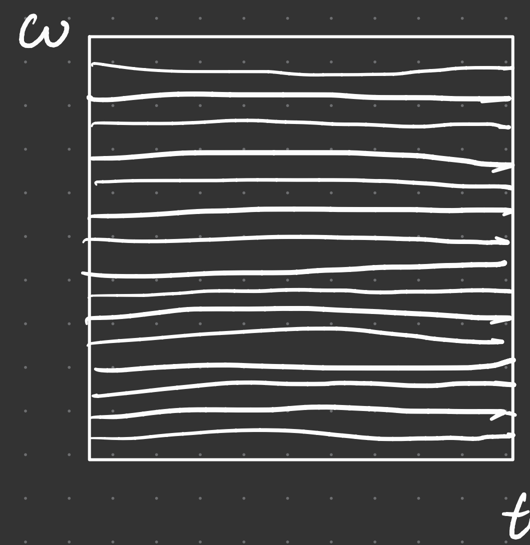
  → wavelets have a logarithmic (base 2) set of bandwidths
    - high-frequency information is captured in short time instants
    - low-frequency information is captured in long time instants

  → wavelets are "blind" to signals that are "locally" polynomial
    - vanishing moments
    - good model for real-life signals
    - great compression performance of wavelets

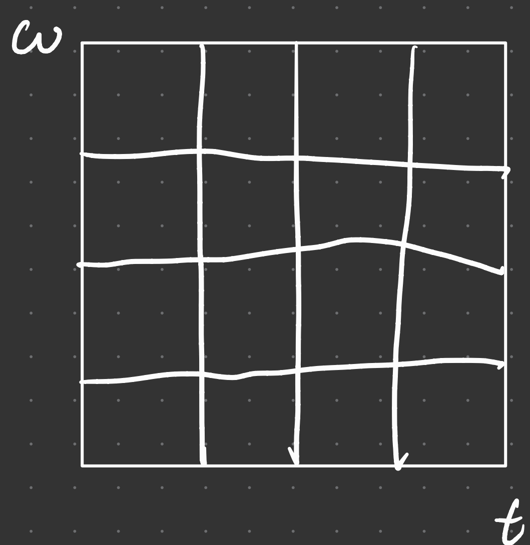
Time - Frequency Tilings



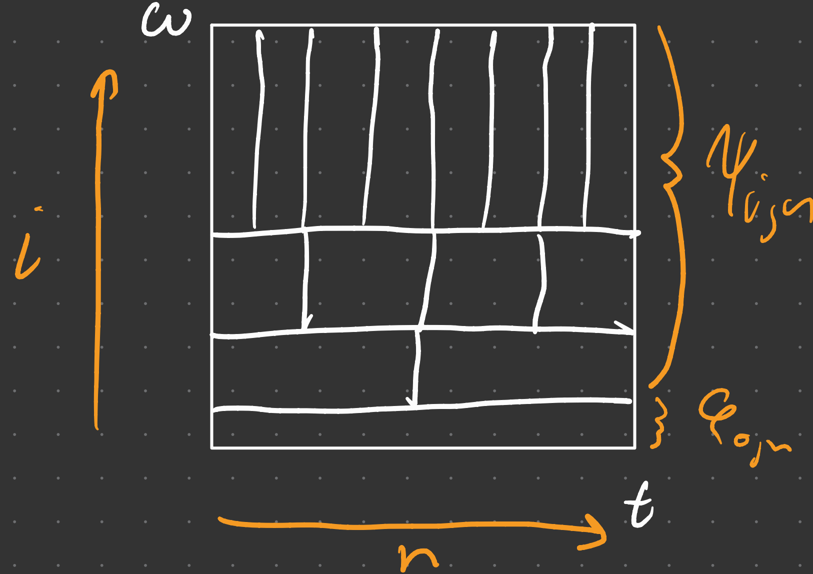
Time-Domain



Frequency-Domain



Spectogram (STFT)



Multiresolution (DWT)

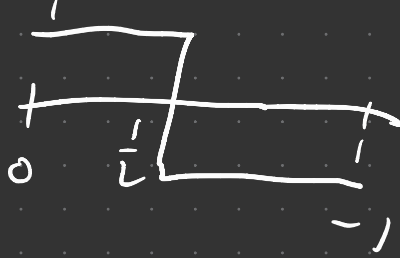
<u>Obs:</u> Time-frequency resolution is dictated by the <u>uncertainty principle</u>.

- The volume of each box is the <u>same</u>.

Haar Wavelets

$\varphi(t)$

$\psi(t)$



- $V_0 = \text{span}\{\varphi(t-n)\}_{n\in\mathbb{Z}}$
- $W_i = \text{span}\{\psi_{i,n}\}_{n\in\mathbb{Z}}$ $\quad \psi_{i,n}(t) = 2^{\frac{i}{2}}\psi(2^i t - n)$
- $L^2(\mathbb{R}) = V_0 \oplus \bigoplus_{i=0}^{\infty} W_i$

$\longrightarrow \{\varphi(t-n)\}_{n\in\mathbb{Z}} \cup \{\psi_{i,n}\}_{n\in\mathbb{Z}}$

is an <u>orthobasis</u> for $L^2(\mathbb{R})$.

<u>Q</u>: What if we just want to analyze signals defined on $[0,1]$?

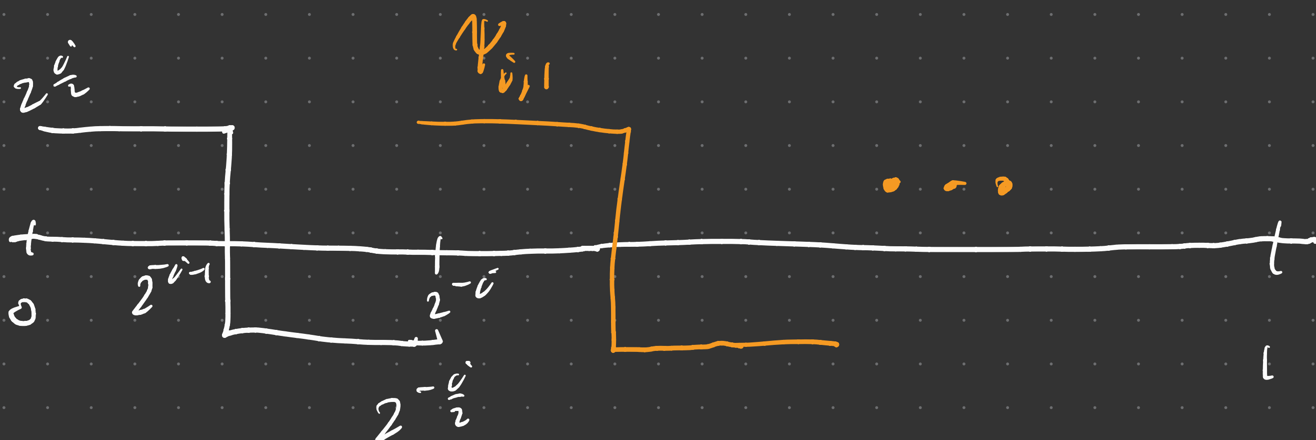
$$L^2[0,1] = \left\{ f:[0,1]\to\mathbb{R} : \int_0^1 |f(t)|^2\,dt < \infty \right\}$$

finite energy signals on $[0,1]$

Exercise: What is the Haar basis for signals defined on $[0,1]$?

Fix a resolution i. How many Haar wavelets intersect $[0,1]$?

$$\psi_{i,0}(t) = 2^{\frac{i}{2}}\psi(2^i t)$$



- Length of each wavelet is 2^{-i}
- Length of $[0,1]$ is 1

$\implies$ # of wavelets is $\frac{1}{2^{-i}} = 2^i$

How many scaling functions intersect $[0,1]$?

$\implies$ only $\varphi(t)$

Haar wavelet basis of $L^2[0,1]$

$$\{e\} \cup \bigcup_{i=0}^{\infty} \bigcup_{n=0}^{2^i-1} \{\psi_{i,n}\}$$ is an ortho basis

for $L^2[0,1]$.

Every $f \in L^2[0,1]$ can be written as

$$f(t) = \underbrace{\langle f, e\rangle e(t)}_{\text{const.}} + \sum_{i=0}^{\infty} \sum_{n=0}^{2^i-1} \langle f, \psi_{i,n}\rangle \psi_{i,n}(t) \qquad t \in [0,1]$$

$e(t) = 1 \quad \forall t \in [0,1]$

$$= \underbrace{\langle f, e\rangle}_{\int_0^1 f(t)\,dt} + \sum_{i=0}^{\infty} \sum_{n=0}^{2^i-1} \underbrace{\langle f, \psi_{i,n}\rangle}_{\theta_{i,n}} \psi_{i,n}(t)$$

Q: What is another ortho basis for $L^2[0,1]$?

A: Fourier basis

Fourier Basis of $L^2[0,1]$

$$\{e^{j2\pi nt}\}_{n\in\mathbb{Z}}$$ is an ortho basis

for $L^2[0,1]$.

Every $f \in L^2[0,1]$ can be written as

$$f(t) = \sum_{n\in\mathbb{Z}} \langle f, e^{j2\pi nt}\rangle e^{j2\pi nt}$$

complex conjugate

$$\langle f, e^{j2\pi nt}\rangle = \underbrace{\int_0^1 f(t)\, e^{-j2\pi nt}\, dt}_{c_n}$$

Fourier coefficients

Fourier series

Exercise: Show that

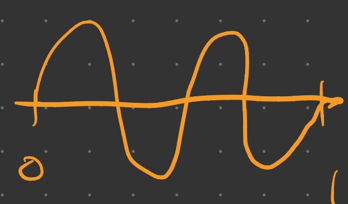
$$\langle e^{j2\pi nt}, e^{j2\pi kt}\rangle = \delta[n-k].$$

$$\int_0^1 e^{j2\pi n t} \, e^{-j2\pi k t} \, dt$$

$$= \int_0^1 e^{j2\pi (n-k) t} \, dt = \delta[n-k]$$

$n = k \Rightarrow e^0 = 1 \Rightarrow \int = 1$

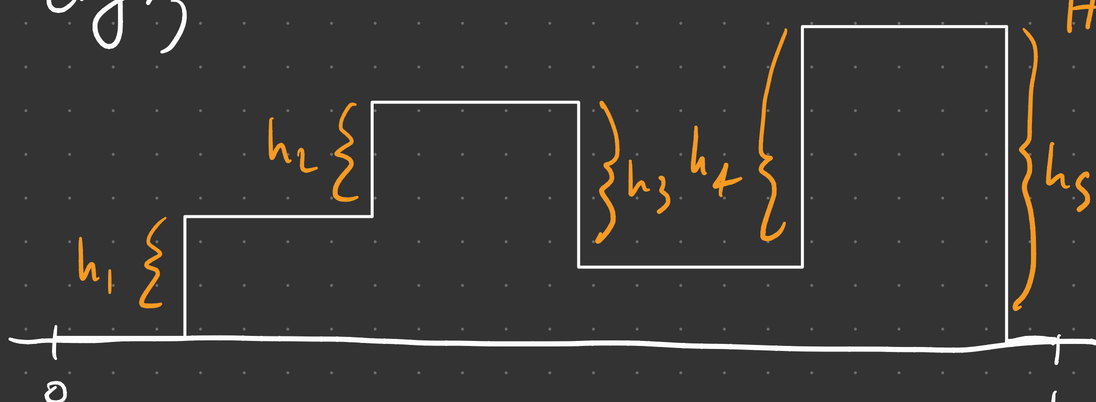
$n \neq k \Rightarrow e^{j2\pi m t} \Rightarrow \int = 0$



Fourier vs. Wavelet

Consider the piecewise constant signal with S pieces

e.g.,



$H = h_1 + h_2 + h_3 + h_4 + h_5$

sum of all the jumps

Exercise: Bound the kth <u>largest</u> Fourier coefficient.

$$|c_n| = \left| \int_0^1 f(t)\, e^{-j2\pi n t}\, dt \right|$$

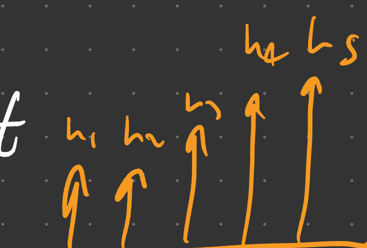
$$\left[\begin{array}{l} u = f(t) \\ dv = e^{-j2\pi n t} \end{array}\right.$$

$$\Rightarrow \left[\begin{array}{l} du = f'(t)\, dt \\ v = \dfrac{e^{-j2\pi n t}}{-j2\pi n} \end{array}\right.$$

$$= \left| uv\Big|_0^1 - \int_0^1 v\, du \right|$$

$$= \left| \int_0^1 f'(t)\, \frac{e^{-j2\pi n t}}{j2\pi n}\, dt \right|$$

$$\le \frac{1}{2\pi |n|} \int_0^1 |f'(t)|\, dt$$



$$= \frac{H}{2\pi |n|}$$

Let $|c_{(1)}| \ge |c_{(2)}| \ge \cdots$ be a <u>non decreasing</u> reordering of the Fourier coeffs.

$$\Rightarrow \boxed{|c_{(k)}| \le H k^{-1}}$$

<u>Exercise</u>: Bound the kth largest wavelet coeff.

Fix the resolution i. Then there are two possible scenarios:

① The support of $\psi_{i,n}$ <u>does not</u> include a jump

$$|\theta_{i,n}| = \left|\int_0^1 f(t)\, \psi_{i,n}(t)\, dt\right| = 0$$

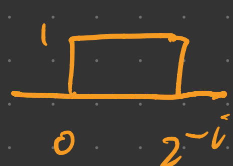
② The support of $\psi_{i,n}$ <u>does</u> include a jump

$$|\theta_{i,n}| = \left|\int_0^1 f(t)\, \psi_{i,n}(t)\, dt\right|$$

$$\leq \int_0^1 \underbrace{|f(t)|}_{\leq H}\, \underbrace{|\psi_{i,n}(t)|}_{2^{\frac{i}{2}}\psi(2^i t - n)}\, dt$$

$$\leq H\, 2^{\frac{i}{2}} \int_0^1 |\psi(2^i t - n)|\, dt$$

$$= H\, 2^{\frac{i}{2}} \int_0^1 |\psi(2^i t)|\, dt$$



$$= H\, 2^{\frac{i}{2}}\, 2^{-i} = H\, 2^{-\frac{i}{2}}$$

Q: How many wavelets will overlap a jump?

A: S per resolution

At resolution i:

$$\sum_{n=0}^{2^i-1} |\theta_{i,n}| \leq SH2^{-\frac{i}{2}}$$

Let $\theta_{i,(n)}$ denote the nth largest coeff at resolution i.

$$SH2^{-\frac{i}{2}} \geq \sum_{n=1}^{2^i} |\theta_{i,(n)}| \geq \sum_{n=1}^{k} |\theta_{i,(k)}|$$

$$\geq k\,|\theta_{i,(k)}|$$

for any $k \in \{1, \dots, 2^i\}$

$$|\theta_{i,(k)}| \leq SH2^{-\frac{i}{2}} k^{-1}$$

At resolution i, the # of coeffs $\geq T$ is

$$K_i \leq \min\{2^i, SH2^{-\frac{i}{2}}T^{-1}\}$$

The number of coeffs $\geq T$ at at all resolutions is

$$K = \sum_{i=0}^{\infty} K_i \leq \sum_{i=0}^{\infty} \min\{2^i, SH2^{-\frac{i}{2}}T^{-1}\}$$

$$= \sum_{i:2^i \leq SH2^{-\frac{i}{2}}T^{-1}} 2^i + \sum_{i:2^i > SH2^{-\frac{i}{2}}T^{-1}} SH2^{-\frac{i}{2}}T^{-1}$$

$$\leq 6 S^{\frac{2}{3}} H^{\frac{2}{3}} T^{-\frac{2}{3}}$$ (check at home)

Let $\theta_{(k)}$ denote the kth largest wavelet coeff: $|\theta_{(1)}| \geq |\theta_{(2)}| \geq \cdots$

Choose $T = |\theta_{(k)}|$

$$\boxed{|\theta_{(k)}| \leq 6^{\frac{3}{2}} S H k^{-\frac{3}{2}}}$$