On Continuous-Domain Inverse Problems with Sparse Superpositions of Decaying Sinusoids as Solutions

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(joint work with Robert D. Nowak)

Continuous-Domain Linear Inverse Problems

- A fundamental problem in science and engineering is to reconstruct a continuous-domain signal $f : \mathbb{R}^d \to \mathbb{R}$ from measurements.

$$y_n = \langle h_n, f \rangle + \varepsilon_n,\ n = 1, \ldots, N$$

- $\mathrm{H}\{f\} = (\langle h_1, f \rangle, \ldots, \langle h_N, f \rangle) \in \mathbb{R}^N$ symbolizes the linear measurement process.
- $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N) \in \mathbb{R}^N$ are perturbation or noise terms, typically zero-mean random variables.
- $\boldsymbol{y} = (y_1, \ldots, y_N) \in \mathbb{R}^N$ denotes the (possibly noisy) data.

Remark

This is an **ill-posed inverse problem**.

Many Real-World Problems are Inverse Problems

$$\boldsymbol{y} = \mathrm{H}\{f\} + \boldsymbol{\varepsilon}$$

- Medical imaging
  - $\Longrightarrow$ Image the interior of a body.
  - $\Longrightarrow$ e.g., in MRI, H corresponds to Fourier-domain measurements.
- Statistics/supervised machine learning
  - $\Longrightarrow$ Learn f from a dataset $\{(\boldsymbol{x}_n, y_n)\}_{n=1}^N$ such that $f(\boldsymbol{x}_n) \approx y_n$.
  - $\Longrightarrow$ $h_n = \delta(\cdot - \boldsymbol{x}_n)$ for some $\boldsymbol{x}_n \in \mathbb{R}^d$.
- Many other real-world problems...

Synthesis Formulation

- Assume *a priori* that f can be **synthesized** by a superposition of atoms from some dictionary, i.e.,

$$f = \sum_{k \in \mathbb{Z}} \alpha_k \varphi_k,$$

where $\{\varphi_k\}_{k \in \mathbb{Z}}$ is a dictionary of atoms, e.g.,
$\Longrightarrow$ wavelets (Donoho, 1998)

- The solution to the inverse problems is synthesized from coefficients that solve

$$\min_{\boldsymbol{\alpha} \in \ell^p(\mathbb{Z})} \left\| \boldsymbol{y} - \mathrm{H}\left\{ \sum_{k \in \mathbb{Z}} \alpha_k \varphi_k \right\} \right\|_2^2 + \lambda \|\boldsymbol{\alpha}\|_p^p,$$

- The choice of $p = 2$ has classically been the common choice, the last few decades have shown that **sparsity** $(p = 1)$ plays a key role in signal reconstruction. (Bruckstein, 2009)
$\Longrightarrow$ Supported by the theory of compressed sensing. (Candès, 2006)

Analysis/Variational Formulation

- The solution to the inverse problem is a solution to the **variational problem**

$$\min_{f \in \mathcal{X}'} \|\boldsymbol{y} - \mathrm{H}\{f\}\|_2^2 + \lambda \|f\|_{\mathcal{X}'}^p,$$

- $\mathcal{X}'$ is an appropriate native space that models the signals to be reconstructed.
- $\|\cdot\|_{\mathcal{X}'}$ is the norm that defines $\mathcal{X}'$.

Sparse Fourier Reconstruction

- In this work we consider the analysis formulation for **sparse** signal reconstruction.
- Consider the **spectral Barron spaces** $\mathscr{B}^s(\mathbb{R}^d)$, $s \geq 0$, which are **Banach spaces** defined by the norm

$$\|f\|_{\mathscr{B}^s(\mathbb{R}^d)} = \|(1 + \|\cdot\|_2)^s F(\cdot)\|_{\mathcal{M}}.$$

$\Longrightarrow$ $F(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\boldsymbol{x}) e^{-\mathrm{j}2\pi\boldsymbol{\xi}^\mathsf{T}\boldsymbol{x}} \, \mathrm{d}\boldsymbol{x}.$

$\Longrightarrow$ The $\mathcal{M}$-norm is a "generalization" of the L^1-norm that can also be applied to distributions such as the Dirac impulse. Morally,

$$\|f\|_{\mathscr{B}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + \|\boldsymbol{\xi}\|_2)^s |F(\boldsymbol{\xi})| \, \mathrm{d}\boldsymbol{\xi}.$$

$\Longrightarrow$ The $\mathcal{M}$-norm is the continuous-domain analogue of the **sparsity-promoting** ℓ^1-norm.

$\Longrightarrow$ These spaces were first studied in the context of approximation theory with neural networks. (Barron, 1993)

Sparse Fourier Reconstruction

Representer Theorem (P. & Nowak, 2022)

Consider the analysis formulation for signal reconstruction where the native space is $\mathscr{B}^s(\mathbb{R}^d)$. Furthermore, suppose that the measurement operator H is weak* continuous on $\mathscr{B}^s(\mathbb{R}^d)$. Then,

$$\mathcal{V} = \underset{f \in \mathscr{B}^s(\mathbb{R}^d)}{\arg\min} \ \|\boldsymbol{y} - \mathrm{H}\{f\}\|_2^2 + \lambda \|f\|_{\mathscr{B}^s(\mathbb{R}^d)},$$

is nonempty, convex, and weak* compact.
The extreme points of $\mathcal{V}$ are given by functions of the form

$$f_{\mathrm{sparse}}(\boldsymbol{x}) = \sum_{k=1}^{K} \alpha_k (1 + \|\boldsymbol{\xi}_k\|_2)^{-s} e^{\mathrm{j}2\pi \boldsymbol{\xi}_k^\mathsf{T} \boldsymbol{x}},$$

where $\boldsymbol{\xi}_k \in \mathbb{R}^d$, $k = 1, \ldots, K$, and $K \leq N$. The weak* closure of the convex hull of these extreme points is the full solution set.

- Why is this kind of result called a representer theorem?

  $\Longrightarrow$ The solution set to the optimization problem is completely characterized by functions with a **finite-dimensional representation** in terms of certain atoms.

- The term representer theorem is commonly used when talking about kernel methods in machine learning.

  $\Longrightarrow$ The notion of a representer theorem is much more general and can be applied to many problems about convex optimization in Banach spaces. (Unser 2021)

Sparse Fourier Reconstruction

$$f_{\text{sparse}}(\boldsymbol{x}) = \sum_{k=1}^{K} \alpha_k (1 + \|\boldsymbol{\xi}_k\|_2)^{-s} e^{\mathrm{j}2\pi \boldsymbol{\xi}_k^\mathsf{T} \boldsymbol{x}},$$

- Solution set of is completely characterized by **sparse** superpositions of **decaying sinusoids**.
  - $\Longrightarrow$ Atoms of solutions: $(1 + \|\boldsymbol{\xi}_k\|_2)^{-s} e^{\mathrm{j}2\pi \boldsymbol{\xi}_k^\mathsf{T} \boldsymbol{x}}$.
  - $\Longrightarrow$ As a function of the frequency variable $\boldsymbol{\xi}_k$, the atoms are decaying sinusoids.
  - $\Longrightarrow$ $\|f_{\text{sparse}}\|_{\mathscr{B}^s(\mathbb{R}^d)} = \|\boldsymbol{\alpha}\|_1$.
  - $\Longrightarrow$ Larger s penalizes high-frequencies due to decay factor $(1 + \|\boldsymbol{\xi}\|_2)^{-s}$.
- Solutions have sparse Fourier transforms.
  - $\Longrightarrow$ Fourier transform is superposition of weighted impulses.
- Condition of weak* continuity of the measurement operator is satisfied by convolution operators whose Fourier transforms decay as $\|\boldsymbol{\xi}\|_2^t$, where $t > -s$, as $\|\boldsymbol{\xi}\|_2 \to \infty$,
  - $\Longrightarrow$ Relatively mild condition.

Numerical Examples

$$\|f\|_{\mathscr{B}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + \|\boldsymbol{\xi}\|_2)^s |F(\boldsymbol{\xi})| \, \mathrm{d}\boldsymbol{\xi}.$$

- Consider digital images ($d = 2$).

  $\implies$ $f \rightsquigarrow \boldsymbol{f}$, where $\boldsymbol{f}$ is an $N \times N$ digital image.
- Replace continuous Fourier transforms with discrete Fourier transforms.

  $\implies$ $\|f\|_{\mathscr{B}^s(\mathbb{R}^2)} \rightsquigarrow \|\boldsymbol{f}\|_{\ell^s}$

  $\implies$ $$\|\boldsymbol{f}\|_{\ell^s} := \sum_{k_1, k_2} \left(1 + \sqrt{\left|\frac{k_1}{N}\right|^2 + \left|\frac{k_2}{N}\right|^2} \right)^s |\boldsymbol{F}[k_1, k_2]|.$$

  $\implies$ $\boldsymbol{F}[k_1, k_2]$ denotes the DFT of $\boldsymbol{f}$.

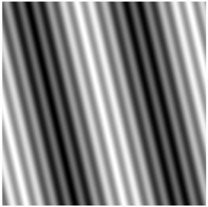
Numerical Examples

We will compare the proposed regularization approach with classical Tikhonov regularization.

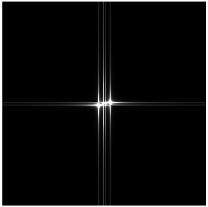
- Proposed approach: $\min_{\boldsymbol{f}} \|\boldsymbol{y} - \mathbf{H}\boldsymbol{f}\|_2^2 + \lambda \|\boldsymbol{f}\|_{\mathscr{b}^s}$
- Tikhonov regularization: $\min_{\boldsymbol{f}} \|\boldsymbol{y} - \mathbf{H}\boldsymbol{f}\|_2^2 + \lambda \|\boldsymbol{f}\|_2^2$
- $\mathbf{H}$ will correspond to convolution with a Gaussian followed by downsampling.
- $\boldsymbol{\varepsilon}$ will be i.i.d. $\mathcal{N}(0,1)$ noise.
- $s = 1$.

Toy Problem

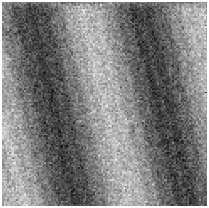
$$\boldsymbol{y} = \mathbf{H}\boldsymbol{f} + \boldsymbol{\varepsilon}$$



512×512 image

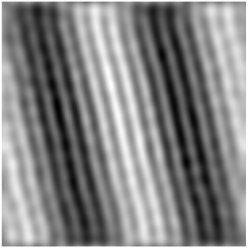


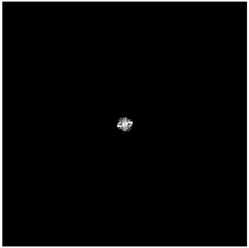
DFT



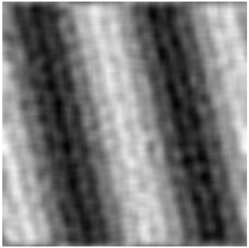
128×128 measurement

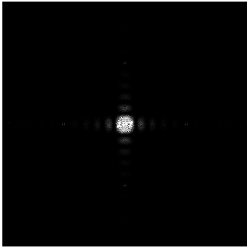
Sparse Fourier Reconstruction



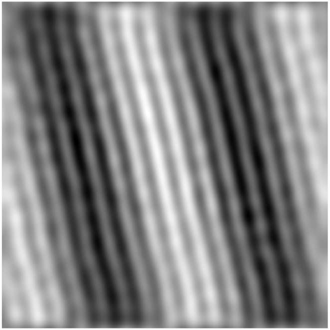


Tikhonov Reconstruction

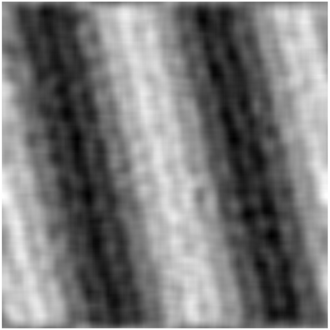




Comparison



Proposed
#coeffs $= 67$



Tikhonov
#Fourier coeffs $= 16384$

Cameraman

512×512 image

128×128 measurement

Cameraman

Proposed
$\#$coeffs $= 180$

Tikhonov
$\#$Fourier coeffs $= 16384$

Concluding Remarks

- Proposed new regularization procedure for sparse signal reconstruction in the **continuous-domain**, adding to a long line of work in this area (Vetterli, 2002), (Eldar, 2005), (Adcock, 2016), (Candés, 2014)

$\Longrightarrow$ Proved a **representer theorem** for the spectral Barron spaces $\mathscr{B}^s(\mathbb{R}^d)$.

- Future work:

$\Longrightarrow$ Comparison of this new approach with existing approaches for sparse signal reconstruction, e.g., wavelet shrinkage and total variation methods.

$\Longrightarrow$ Quantify conditions for **exact recovery** for signals in $f \in \mathscr{B}^s(\mathbb{R}^d)$ from measurements.
When $s = 0$ the problem has been solved under the name "off-the-grid" compressed sensing. (Tang, 2013), (Candés, 2014)